# SF1684 Algebra and Geometry

Lecture 6

Similar matrices, diagonalization, quadratic forms, general vector spaces

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# Outline

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# Matrices with respect to bases

Recall that every linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ has an associated standard matrix

$$[T] = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{bmatrix}$$

with the property that $T(\mathbf{x}) = [T]\mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^n$.

## Matrix of a linear operator with respect to a basis

Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator and let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for $\mathbb{R}^n$. Define the $n \times n$ matrix

$$A = \begin{bmatrix} [T(\mathbf{v}_1)]_B & [T(\mathbf{v}_2)]_B & \cdots & [T(\mathbf{v}_n)]_B \end{bmatrix}.$$

Then

$$[T(\mathbf{x})]_B = A[\mathbf{x}]_B \quad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

Moreover, $A$ is the only matrix with this properties. It is called the **matrix for $T$ with respect to the basis $B$** and is denoted by $[T]_B$.

## Warning

The matrix for $T$ with respect to the standard basis $S$ for $\mathbb{R}^n$ is the same as the standard matrix for $T$ and is denoted by $[T]$.

## Matrix of a linear transformation with respect to pair of bases

Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $B' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ be bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Define the $m \times n$ matrix

$$A = \begin{bmatrix} [T(\mathbf{v}_1)]_{B'} & [T(\mathbf{v}_2)]_{B'} & \cdots & [T(\mathbf{v}_n)]_{B'} \end{bmatrix}.$$

Then

$$[T(\mathbf{x})]_{B'} = A[\mathbf{x}]_B \quad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

Moreover, $A$ is the only matrix with this properties. It is called the **matrix for $T$ with respect to the bases $B$ and $B'$** and is denoted by $[T]_{B',B}$.

## Examples

See Examples 1–7 on pages 445–453 of the textbook.

# Changing bases

## Matrices of a operator in different bases

Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator and let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $B' = \{\mathbf{v}'_1, \mathbf{v}'_2, \ldots, \mathbf{v}'_n\}$ be bases for $\mathbb{R}^n$. Then

$$[T]_{B'} = P[T]_B P^{-1},$$

where $P$ is the transition matrix from $B$ to $B'$

$$P = P_{B\to B'} = \begin{bmatrix} [\mathbf{v}_1]_{B'} & [\mathbf{v}_2]_{B'} & \cdots & [\mathbf{v}_n]_{B'} \end{bmatrix}.$$

If $B$ and $B'$ are **orthonormal** bases, then $P$ is orthogonal, so that

$$[T]_{B'} = P[T]_B P^T.$$

In the case $B' = S$ is the standard basis for $\mathbb{R}^n$, then $[T]_{B'} = [T]$ and $P = P_{B\to S} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$.

# Outline

# Similar matrices

## Definition

Two $n \times n$ matrices $C$ and $A$ that are related by $C = P^{-1}AP$ where $P$ is an invertible matrix are said to be similar.

## Similarity from an operator point of view

Two square matrices are similar if and only if there exist bases with respect to which they represent the same linear operator.

## Similarity invariants

Similar matrices have the same determinant, rank, nullity, trace, characteristic polynomial, and eigenvalues.

# Eigenvalues and eigenvectors of similar matrices

Let $A$ be an $n \times n$ matrix and $\lambda$ an eigenvalue of $A$.

## Algebraic multiplicity of eigenvalues

The multiplicity of the root $\lambda$ of the characteristic polynomial of $A$ is called the algebraic multiplicity of $\lambda$.

Recall that the eigenspace of $A$ corresponding to $\lambda$ is the solution space of the linear system $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

## Geometric multiplicity of eigenvalues

The dimension of the eigenspace corresponding to $\lambda$ is called the geometric multiplicity of $\lambda$.

## Eigenvalues of similar matrices

The eigenvalues of similar matrices have the same algebraic and geometric multiplicities.

## Eigenvectors of similar matrices

Suppose that $C = P^{-1}AP$ and $\lambda$ is an eigenvalue of $A$ and $C$.

i. If $\mathbf{x}$ is an eigenvector of $C$ corresponding to $\lambda$, then $P\mathbf{x}$ is an eigenvector of $A$ corresponding to $\lambda$.
ii. If $\mathbf{x}$ is an eigenvector of $A$ corresponding to $\lambda$, then $P^{-1}\mathbf{x}$ is an eigenvector of $C$ corresponding to $\lambda$.

## Examples

See Examples 1–3 on pages 457–459 of the textbook.

# Diagonalization

## The diagonalization problem

Given a square matrix $A$, does there exist an invertible matrix $P$ for which $P^{-1}AP$ is a diagonal matrix? If so, how do we find $P$? If such a $P$ exists, then $A$ is said to be **diagonalizable** and $P$ is said to **diagonalize** $A$.

## Diagonalizable matrix

An $n \times n$ matrix $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$. In this case, the matrix $P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \ldots & \mathbf{p}_n \end{bmatrix}$ is invertible and

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix},$$

where $\lambda_i$ is the eigenvalue of $A$ corresponding to $\mathbf{p}_i$. These eigenvectors must then form a basis for $\mathbb{R}^n$.

## Linearly independent eigenvectors

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are eigenvectors of $A$ corresponding to **distinct** eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, then they are linearly independent.

## Corollary

An $n \times n$ matrix with $n$ distinct real eigenvalues is diagonalizable.

## Warning

It is possible for an $n \times n$ matrix to be diagonalizable without having $n$ distinct eigenvalues.

## Diagonalizability and geometric multiplicity

An $n \times n$ matrix $A$ is diagonalizable if and only if the sum of the geometric multiplicities of its eigenvalues is $n$.

## Method for diagonalizing $A$

1. Find all eigenvalues of $A$.
2. For each eigenvalue $\lambda$, compute the canonical solutions for the solution space (eigenspace) of $(\lambda I - A)\mathbf{x} = \mathbf{0}$. If there are $n$ canonical solutions in total, say $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$, then $A$ is diagonalizable.
3. Form the matrix $P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \ldots & \mathbf{p}_n \end{bmatrix}$.
4. The matrix $P^{-1}AP$ will be diagonal

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

## Examples

See Examples 1–7 on pages 457–464 of the textbook.

## Relationship between algebraic and geometric multiplicity

Let $A$ be an square matrix

i. For each eigenvalue $\lambda$ of $A$, the geometric multiplicity of $\lambda$ is less than or equal to its algebraic multiplicity.
ii. $A$ is diagonalizable if and only if for every eigenvalue $\lambda$ of $A$, the geometric multiplicity of $\lambda$ is equal to its algebraic multiplicity.

## Unifying theorem on diagonalizability

If $A$ is an $n \times n$ matrix, then the following statements are equivalent

a. $A$ is diagonalizable.
b. $A$ has $n$ linearly independent eigenvectors.
c. $\mathbb{R}^n$ has a basis consisting of eigenvectors of $A$.
d. The sum of the geometric multiplicities of the eigenvalues of $A$ is $n$.
e. The geometric multiplicity of each eigenvalue of $A$ is the same as the algebraic multiplicity.

# Outline

# Orthogonally similar matrices

## Definition

Two $n \times n$ matrices $C$ and $A$ that are related by $C = P^T AP$ where $P$ is an orthogonal matrix are said to be orthogonally similar.

## Orthogonal similarity from an operator point of view

Two square matrices are orthogonally similar if and only if there exist orthonormal bases with respect to which they represent the same linear operator.

## The orthogonal diagonalization problem

Given a square matrix $A$, does there exist an orthogonal matrix $P$ for which $P^T AP$ is a diagonal matrix? If so, how do we find $P$? If such a $P$ exists, then $A$ is said to be **orthogonally diagonalizable** and $P$ is said to **orthogonally diagonalize** $A$.

# Orthogonal diagonalization

## Orthogonally diagonalizable matrix

An $n \times n$ matrix $A$ is orthogonally diagonalizable if and only if $A$ has an orthonormal set of $n$ eigenvectors $\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n\}$. In this case, the matrix $P = \begin{bmatrix}\mathbf{p}_1 & \mathbf{p}_2 & \ldots & \mathbf{p}_n\end{bmatrix}$ is orthogonal and

$$P^T A P = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix},$$

where $\lambda_i$ is the eigenvalue of $A$ corresponding to $\mathbf{p}_i$. These eigenvectors must then form an orthonormal basis for $\mathbb{R}^n$.

## Diagonalizability of symmetric matrix

A matrix which is orthogonally diagonalizable is symmetric. Conversely, if $A$ is a symmetric matrix, then

- i $A$ is orthogonally diagonalizable.
- ii The eigenvectors of $A$ corresponding to **distinct** eigenvalues are orthogonal.

## Method for orthogonally diagonalizing a symmetric matrix $A$

1. Find all eigenvalues $\lambda$ of $A$, then find a basis for the eigenspace corresponding to each $\lambda$. There will be $n$ vectors in total.
2. Apply the Gram-Schmidt process to **each** of these bases to produce orthonormal bases for the eigenspaces.
3. Form the matrix $P = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{bmatrix}$ whose columns are the vectors constructed in Step 2.
4. The matrix $P^TAP$ will be diagonal

$$P^TAP = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

## Examples

See Example 1 on page 470 of the textbook.

# Spectral decomposition

## Spectral decomposition

If $A$ is an $n \times n$ symmetric matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding orthonormal eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$, then

$$\begin{aligned} A = PDP^T &= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix} \\ &= \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T. \end{aligned}$$

This formula is called a **spectral decomposition of $A$** or an **eigenvalue decomposition of $A$**.

## Example

See Example 2 on page 472 of the textbook.

# Power of a matrix

## Power of a diagonalizable matrix

If $A$ is diagonalizable and $P^{-1}AP = D$ is a diagonal matrix, then, for any positive integer $k$,

$$P^{-1}A^kP = D^k \quad \text{which we can rewrite as} \quad A^k = PD^kP^{-1}.$$

## Cayley-Hamilton theorem

Every square matrix satisfies its characteristic equation: if $A$ is an $n \times n$ matrix whose characteristic equation is

$$\lambda^n + c_1\lambda^{n-1} + \cdots + c_n = 0$$

then

$$A^n + c_1A^{n-1} + \cdots + c_nI_n = 0.$$

## Examples

See Examples 3, 4 on pages 473, 475 of the textbook.

# Outline

# Quadratic forms

## Definition

A quadratic form $Q$ is a polynomial in several variables with coefficients in $\mathbb{R}$, and such that each term has total degree $2$, that is,

$$Q(\mathbf{x}) = \sum_{1 \leq i,j \leq n} a_{ij} x_i x_j, \quad \text{with } a_{ij} \in \mathbb{R}.$$

## Quadratic form in matrix notation

If $A$ is an $n \times n$ symmetric matrix and $x$ is an $n \times 1$ column vector of variables, then the scalar

$$Q_A(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$$

is called the **quadratic form associated with $A$**.

In the case where $A$ is a diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

$$Q_A(\mathbf{x}) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2.$$

## Express the quadratic form as $\mathbf{x}^T A \mathbf{x}$ where $A$ is symmetric

General quadratic form on $\mathbb{R}^2$

$$a_1 x_1^2 + a_2 x_2^2 + a_3 x_1 x_2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_1 & a_3/2 \\ a_3/2 & a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

General quadratic form on $\mathbb{R}^3$

$$\begin{aligned} & a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_1 x_2 + a_5 x_1 x_3 + a_6 x_2 x_3 \\ & \qquad = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_1 & a_4/2 & a_5/2 \\ a_4/2 & a_2 & a_6/2 \\ a_5/2 & a_6/2 & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \end{aligned}$$

# Orthogonal change of variable

## Principal axes theorem

If $A$ is a symmetric matrix, it can be diagonalized as $D = P^T AP$, where $P$ is the orthogonal matrix of eigenvectors of $A$. Suppose that $\mathbf{x} = P\mathbf{y}$, then

$$\begin{aligned} Q_A(\mathbf{x}) &= \mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T P^T AP\mathbf{y} = \mathbf{y}^T D\mathbf{y} \\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2. \end{aligned}$$

## Example

See Example 2 on page 483 of the textbook.

# Positive definite

## Positive definite quadratic form

A symmetric matrix $A$ and its quadratic form $Q_A(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ are said to be

1. **positive definite** if $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$.
2. **negative definite** if $\mathbf{x}^T A\mathbf{x} < 0$ for all $\mathbf{x} \neq \mathbf{0}$.
3. **indefinite** if $\mathbf{x}^T A\mathbf{x}$ has both positive and negative values.

There are other two possibilities: $A$ is called **positive semidefinite** if $\mathbf{x}^T A\mathbf{x} \geq 0$ and **negative semidefinite** if $\mathbf{x}^T A\mathbf{x} \leq 0$.

## Determining whether a matrix is positive definite

If $A$ is a symmetric matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

1. $A$ is positive definite if and only if $\lambda_i > 0$ for all $1 \leq i \leq n$.
2. $A$ is negative definite if and only if $\lambda_i < 0$ for all $1 \leq i \leq n$.
3. $A$ is indefinite if and only if $\lambda_i > 0$ and $\lambda_j < 0$ for some $1 \leq i, j \leq n$.

## Principal submatrix

The $k$th principal submatrix of an $n \times n$ matrix $A$ is the $k \times k$ submatrix consisting of the first $k$ rows and columns of $A$.

## Identifying positive definite matrix

A symmetric $A$ is positive definite if and only if the determinant of every principal submatrix is positive.

## Factorization of positive definite matrix

If $A$ is a symmetric matrix, then the following are equivalent

a. $A$ is positive definite.
b. There is a symmetric positive definite matrix $B$ such that $A = B^2$.
c. There is an invertible matrix $C$ such that $A = C^T C$.

## Examples

See Examples 4–6 on pages 489–491 of the textbook.

# Geometric interpretation of quadratic forms in $\mathbb{R}^2$

## Conic sections

Consider the quadratic form $Q_A(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ in $\mathbb{R}^2$ and let $k \neq 0$ be a constant. Then the equation $\mathbf{x}^T A\mathbf{x} = k$ represents a **central conic**. These include circles, ellipses, and hyperbolas, but not parabolas.

1. If $A$ is diagonal, then the conic is symmetric about both coordinates axes (called standard position).
2. If $A$ is not diagonal, then the conic is rotated out of standard position. We can use the Principal Axes Theorem to put the conic in standard position.

## Example

See Example 3 on page 487 of the textbook.

## Classifying conic sections using eigenvalues

Let $A$ be a symmetric $2 \times 2$ matrix and $\lambda_1, \lambda_2$ its eigenvalues. Then the equation $\mathbf{x}^T A \mathbf{x} = 1$

1. represents an ellipse if $\lambda_1 > 0$ and $\lambda_2 > 0$ ($A$ is positive definite).
2. has no graph if $\lambda_1 < 0$ and $\lambda_2 < 0$ ($A$ is negative definite).
3. represents an hyperbola if $\lambda_1 \lambda_2 < 0$ ($A$ is indefinite).

# Outline

# General vector spaces

## Vector space axioms

Let $V$ be a nonempty set of objects on which the operations of addition and scalar multiplication are defined. By **addition**, we mean a rule for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in $V$ an object $\mathbf{u}+\mathbf{v}$, called the sum of $\mathbf{u}$ and $\mathbf{v}$; by **scalar multiplication**, we mean a rule for associating with each scalar $k$ and each object $\mathbf{u}$ in $V$ an object $k\mathbf{u}$, called the scalar multiple of $\mathbf{u}$ by $k$. If the following axioms are satisfied by all objects $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $V$ and all scalars $k$, $l$, then we call $V$ a **vector space** and we call the objects in $V$ **vectors**:

1. If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u}+\mathbf{v} \in V$. We say that $V$ is **closed under addition.**
2. $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$
4. $V$ contains an object $\mathbf{0}$, called the **zero vector**, which satisfies $\mathbf{u}+\mathbf{0} = \mathbf{u}$ for every vector $\mathbf{u} \in V$.
5. For each $\mathbf{u} \in V$, there exists an object $-\mathbf{u}$, called the **negative of $\mathbf{u}$**, such that $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$.

## Vector space axioms (cont.)

6. If $\mathbf{u} \in V$, and $k$ is a scalar, then $k\mathbf{u} \in V$. We say that $V$ is **closed under scalar multiplication**.
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$
9. $k(l\mathbf{u}) = (kl)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

## Properties of scalar multiplication

If $\mathbf{v} \in V$ and $k$ is a scalar, then

- i $0\mathbf{v} = \mathbf{0}$
- ii $k\mathbf{0} = \mathbf{0}$
- iii $(-1)\mathbf{v} = -\mathbf{v}$

## Warning

We call $V$ a real vector space, if the scalars are real numbers. The scalars could actually belong to any field such as $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$.

## Subspace

A nonempty subset $W$ of a vector space $V$ is called a subspace if $W$ is closed under scalar multiplication and addition.

## Linear independence, spanning, basis

The definitions of linear combination, linear independence, spanning, and basis carry over to general vector spaces.

## Dimension

A vector space $V$ is **finite-dimensional** if it has a basis with finitely many vectors, and **infinite-dimensional** otherwise. If $V$ is finite-dimensional, then the dimension of $V$ is the number of vectors in any basis; otherwise the dimension of $V$ is infinite. In addition, the zero vector space $V = \{\mathbf{0}\}$ is defined to be have dimension $0$.

## Examples

- The most important examples of finite-dimensional vector spaces are $n$-dimensional Euclidean space $\mathbb{R}^n$, and $n$-dimensional complex space $\mathbb{C}^n$.
- The set of all $m \times n$ matrices with real entries under the standard operations of matrix addition and matrix scalar multiplication is a finite-dimensional vector space, which we denote by $M_{mn}$.
- The set of real-valued functions defined on the entire real line is a infinite-dimensional vector space under the ordinary addition and scalar multiplication of functions, which we denote by $F(-\infty, \infty)$.
- The set $C(-\infty, \infty)$ of all real-valued continuous functions defined on the entire real line is a subspace of $F(-\infty, \infty)$.
- The set $P_n$ of all polynomials of degree $n$ or less is a finite-dimensional subspace of $C(-\infty, \infty)$.

# Outline

# Linear transformations

## Linear transformation

Let $V$ and $W$ be vector spaces. A function $T: V \to W$ is called a linear transformation if

1. $T(c\mathbf{u}) = cT(\mathbf{u})$ and
2. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$ and all scalars $c$.

In the special case where $V = W$, $T$ is called a **linear operator** on $V$.

## Some properties

If $T$ is a linear transformation then

- i $T(\mathbf{0}) = \mathbf{0}$.
- ii $T(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k) = c_1T(\mathbf{u}_1) + c_2T(\mathbf{u}_2) + \cdots + c_kT(\mathbf{u}_k)$.

## Kernel and range, one-to-one, onto

The definitions of kernel and range, one-to-one, onto transformations carry over to general vector spaces.

# Isomorphism

## Definition

A linear transformation $T\colon V \to W$ is called an **isomorphism** if it is one-to-one and onto, and we say a vector space $V$ is **isomorphic** to $W$ if there exists an isomorphism from $V$ onto $W$.

Let $V$ be a real $n$-dimensional vector space and $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be any basis for $V$. Suppose that $\mathbf{u}$ is a vector in $V$ and $\mathbf{u} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$, then the coordinate vector $(\mathbf{u})_B = (k_1, k_2, \ldots, k_n)$ is an element of $\mathbb{R}^n$. Moreover,

## Theorem

The transformation $T(\mathbf{u}) = (\mathbf{u})_B$ is an isomorphism from $V$ onto $\mathbb{R}^n$.

Thus, we have

**Theorem**

Every real $n$-dimensional vector space is isomorphic to $\mathbb{R}^n$ and hence has the same algebraic structure as $\mathbb{R}^n$.

**Examples**

See Examples 18–20 on pages 590–591 of the textbook.