SF1684 Algebra and Geometry

Lecture 3

Determinants, eigenvalues and eigenvectors

Bao Nguyen

Outline

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Definitions

Permutations

A permutation of n elements $\{1, 2, \dots, n\}$ is a rearrangement of these elements in a specific order, say $\sigma = \{\sigma_1, \sigma_2, \dots, \sigma_n\}$. There are $n! = n(n-1)\cdots 1$ different permutations of $\{1, 2, \dots, n\}$.

An **inversion** is a pair $i < j$ such that $\sigma_i > \sigma_j$. The sign of σ is defined as

$$\operatorname{sgn}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ has an even number of inversions,} \\ -1 & \text{if } \sigma \text{ has an odd number of inversions.} \end{cases}$$

Definition of determinants

The determinant of an $n \times n$ matrix A is

$$\det(A) = \sum_{\sigma} \operatorname{sgn}(\sigma) a_{1\sigma_1} a_{2\sigma_2} \cdots a_{n\sigma_n},$$

where σ runs over all permutations of $\{1, 2, \dots, n\}$.

Determinants of 2×2 and 3×3 matrices

The 2×2 case

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

The 3×3 case

The 3×3 determinant can be written in terms of 2×2 determinants

$$\begin{aligned}
\det(A) &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
&= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\
&= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1) \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}
\end{aligned}$$

Definition using expansion by cofactors

Let $A = [a_{ij}]$ be an $n \times n$ matrix.

Minors and cofactors

The A_{ij} **submatrix** of A is the $(n-1) \times (n-1)$ matrix obtained from A by deleting its row and column containing a_{ij} (that is, row i and column j). Then $M_{ij} = \det(A_{ij})$ is called the **minor** and $C_{ij} = (-1)^{i+j} M_{ij}$ the **cofactor** of entry a_{ij}.

Recursive definition of determinants

If A is a 1×1 matrix then $\det(A) = a_{11}$, else

$$\det(A) = \sum_{j=1}^{n} a_{1j}(-1)^{1+j} \det(A_{1j}) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

Cofactor expansions

The determinant of A can be obtained by a cofactor expansion along any row or any column. In particular, the expansion of the determinant along the ith row of A is

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$

The expansion of the determinant along the jth column of A is

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

Find determinant using cofactor expansions

The cofactor expansion rewrites the determinant of a big matrix in terms of the determinants of smaller matrices. This method is especially applicable if a matrix has a row or a column with **many zeros**. Then we expand the determinant along this row or column.

Example

Compute $\det(A)$ where $A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 3 & 1 & 0 & 2 \\ 2 & -2 & 0 & 4 \\ 1 & 3 & 1 & -2 \end{bmatrix}$.

Solution We use the expansion along the third column (omitting zero terms)

$$\begin{vmatrix} 1 & 0 & 2 & -1 \\ 3 & 1 & 0 & 2 \\ 2 & -2 & 0 & 4 \\ 1 & 3 & 1 & -2 \end{vmatrix} = 2(-1)^{1+3} \begin{vmatrix} 3 & 1 & 2 \\ 2 & -2 & 4 \\ 1 & 3 & -2 \end{vmatrix} + 1(-1)^{4+3} \begin{vmatrix} 1 & 0 & -1 \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix}$$

$$= 2(-1)^{1+3}0 + 1(-1)^{4+3}16 = -16.$$

Special cases

Matrix with one row or one column of zeros

If A has one row or one column of zeros, then $\det(A) = 0$.

Triangular matrix

If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then $\det(A)$ is the product of the diagonal entries of A

$$\det(A) = a_{11}a_{22}\cdots a_{nn}.$$

Elementary row operations and determinants

Determinant of transpose

$$\det(A^T) = \det(A).$$

Effect of elementary row operations on determinant

B is obtained from A by performing one elementary row operation.

$A \xrightarrow{R_i \leftrightarrow R_j} B$	$\det(B) = -\det(A)$
$A \xrightarrow{R_i \mapsto cR_i} B$ for $c \neq 0$	$\det(B) = c\det(A)$
$A \xrightarrow{R_i \mapsto R_i + cR_j} B$ for $i \neq j$	$\det(B) = \det(A)$

Remark

We have similar conclusions for **elementary column operations**, since a column operation on A has the same effect as the corresponding row operation on A^T.

Find determinant using Gaussian elimination

We reduce the matrix to row echelon form, keeping track of how the determinant changes. The determinant of row echelon form, which is a triangular matrix, is the product of its diagonal entries.

Example

Compute $\det(A)$ where $A = \begin{bmatrix} 2 & 0 & 4 \\ 3 & 0 & 1 \\ 1 & 3 & 2 \end{bmatrix}$.

Solution

$$\begin{vmatrix} 2 & 0 & 4 \\ 3 & 0 & 1 \\ 1 & 3 & 2 \end{vmatrix} \overset{R_1 \mapsto (1/2)R_1}{=} 2 \begin{vmatrix} 1 & 0 & 2 \\ 3 & 0 & 1 \\ 1 & 3 & 2 \end{vmatrix} \overset{\substack{R_2 \mapsto R_2 + (-3)R_1 \\ R_3 \mapsto R_3 + (-1)R_1}}{=} 2 \begin{vmatrix} 1 & 0 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 0 \end{vmatrix}$$

$$\overset{R_2 \leftrightarrow R_3}{=} 2(-1) \begin{vmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{vmatrix} = 2(-1)(-15) = 30.$$

The last matrix is triangular, so we can stop the process and compute its determinant.

Properties of determinants

Theorem

1. If A has two identical rows or columns, then $\det(A) = 0$.
2. If A has two proportional rows or columns, then $\det(A) = 0$.
3. $\det(cA) = c^n \det(A)$ for $c \in \mathbb{R}$.

Since the effect of elementary row operations on determinant, the determinant of a square matrix A and the determinant of its reduced echelon form R are both zero or both nonzero. Thus, we have the result

Determinant and invertibility

A is invertible if and only if $\det(A) \neq 0$.

Determinant of product

If A and B are square matrices of the same size, then

$$\det(AB) = \det(A)\det(B).$$

As a consequence, $\det(A^m) = \big(\det(A)\big)^m$ for $m \in \mathbb{N}$.

Determinant of inverse

If A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Fundamental theorem of invertible matrices (cont.)

If A is an $n \times n$ matrix, then the following statements are equivalent

a. The reduced echelon form of A is I_n.
b. A is a product of elementary matrices.
c. A is invertible.
d. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^n$.
f. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^n$.
g. The column vectors of A are linearly independent.
h. The row vectors of A are linearly independent.
i. $\det(A) \neq 0$.

Cramer's rule

Adjoint matrix

The matrix formed by all of the cofactors C_{ij} of the entries a_{ij} is called the matrix of cofactors (cofactor matrix) from A

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}.$$

The transpose of this matrix is called the adjoint (or adjugate) of A and is denoted by $\mathrm{adj}(A)$

$$\mathrm{adj}(A) = C^T.$$

Inverse formula

If A is invertible, then

$$A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A).$$

Cramer's rule

Let A be an $n \times n$ matrix and $\mathbf{b}$ be an $n \times 1$ column vector. The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if $\det(A) \neq 0$, in which case the solution is

$$\mathbf{x} = \left(\frac{\det(A_1)}{\det(A)}, \frac{\det(A_2)}{\det(A)}, \dots, \frac{\det(A_n)}{\det(A)} \right),$$

where A_j is the matrix formed by replacing the jth column of A by $\mathbf{b}$.

The Cramer's rule is useful when solving linear systems with symbolic coefficients.

Example

Solve for x and y in terms of x' and y'

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}, \quad \theta, x, y, x', y' \in \mathbb{R}.$$

Solution The determinant of the coefficient matrix is

$$\begin{vmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{vmatrix} = \cos^2(\theta) + \sin^2(\theta) = 1.$$

Thus, Cramer's rule yields

$$x = \begin{vmatrix} x' & -\sin(\theta) \\ y' & \cos(\theta) \end{vmatrix} = x' \cos(\theta) + y' \sin(\theta)$$

$$\text{and} \quad y = \begin{vmatrix} \cos(\theta) & x' \\ \sin(\theta) & y' \end{vmatrix} = y' \cos(\theta) - x' \sin(\theta).$$

Determinants as area or volume

Theorem

1. If A is a 2×2 matrix, the area of the parallelogram determined by the column vectors of A is $|\det(A)|$.
2. If A is a 3×3 matrix, the volume of the parallelepiped determined by the column vectors of A is $|\det(A)|$.

Example

Find the area of the triangle with vertices $A(-1,-2), B(0,4)$ and $C(3,0)$.

Solution The area of the triangle is half of the parallelogram that has adjacent sides $\overrightarrow{AB} = (1,6)$ and $\overrightarrow{AC} = (4,2)$. Thus

$$\text{area } \triangle ABC = \frac{1}{2}\left|\det\begin{bmatrix} 1 & 4 \\ 6 & 2 \end{bmatrix}\right| = \frac{1}{2}\,|-22| = 11.$$

Outline

Cross products

Definition

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^3$, then the cross product of $\mathbf{u}$ with $\mathbf{v}$ is the vector in $\mathbb{R}^3$ defined by

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right) \\ &= (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).\end{aligned}$$

We can write $\mathbf{u} \times \mathbf{v}$ in the form of a 3×3 determinant as

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{e}_1 - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{e}_2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{e}_3,$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are standard unit vectors in $\mathbb{R}^3$.

Theorem

The vector $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, that is $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ and $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$.

Algebraic properties

If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and $k \in \mathbb{R}$, then

- i $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
- ii $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
- iii $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$
- iv $k(\mathbf{u} \times \mathbf{x}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$
- v $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$
- vi $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

Warning

The cross product is neither commutative nor associative.

Direction of cross product

The direction of $\mathbf{u} \times \mathbf{v}$ is given by the **right-hand rule**: If the fingers of your right hand curl in the direction of rotation (through an angle less than $180°$) from $\mathbf{u}$ to $\mathbf{v}$, then your thumb points in the direction of $\mathbf{u} \times \mathbf{v}$.

Length of cross product

If θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$, then

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin(\theta),$$

which is also equal to the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.

Corollary

Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

Outline

Eigenvalues and eigenvectors

Definition

Let A be an $n \times n$ matrix. A scalar λ is called an eigenvalue of A if there is a **nonzero** vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Such an $\mathbf{x}$ is called an eigenvector of A corresponding to λ.

Eigenspace

The eigenspace (the set of all eigenvectors) of A corresponding to λ is the solution space of the homogeneous linear system $(\lambda I_n - A)\mathbf{x} = \mathbf{0}$.

Finding eigenvalues

The linear system $(\lambda I_n - A)\mathbf{x} = \mathbf{0}$ has nontrivial solutions if and only if $\det(\lambda I_n - A) = 0$. Thus, we have the result

Theorem

The following statements are equivalent

a. λ is an eigenvalue of A.
b. λ is a solution of the equation $\det(\lambda I_n - A) = 0$.
c. The linear system $(\lambda I_n - A)\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

Characteristic polynomial

The expression $p(\lambda) = \det(\lambda I_n - A)$ is a polynomial in the variable λ of degree n. We call this polynomial the characteristic polynomial of A.

Remark

The matrix A has exactly n eigenvalues (not necessarily distinct and could be complex numbers).

Fundamental theorem of invertible matrices (cont.)

If A is an $n \times n$ matrix, then the following statements are equivalent

a. The reduced echelon form of A is I_n.
b. A is a product of elementary matrices.
c. A is invertible.
d. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^n$.
f. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^n$.
g. The column vectors of A are linearly independent.
h. The row vectors of A are linearly independent.
i. $\det(A) \neq 0$.
j. $\lambda = 0$ is not an eigenvalue of A.

Special cases

Eigenvalues of triangular matrix

If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal) with diagonal entries $a_{11}, a_{22}, \ldots, a_{nn}$, then the characteristic polynomial of A is

$$\det(\lambda I_n - A) = (\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn}).$$

It implies that the eigenvalues of A are

$$\lambda_1 = a_{11}, \quad \lambda_2 = a_{22}, \quad \ldots, \quad \lambda_n = a_{nn}.$$

Eigenvalues of matrix power

If λ is an eigenvalue of a matrix A and $\mathbf{x}$ is a corresponding eigenvector, then the following holds for any positive integer k

$$A^k \mathbf{x} = \lambda^k \mathbf{x}.$$

Consequently, λ^k is an eigenvalue of A^k with corresponding eigenvector $\mathbf{x}$.

Factoring the characteristic polynomial

Let A be an $n \times n$ matrix, and let $p(\lambda) = \det(\lambda I_n - A)$ be its characteristic polynomial.

Algebraic multiplicity of eigenvalues

The multiplicity of a root λ of $p(\lambda)$ is called the algebraic multiplicity of the eigenvalue λ.

Theorem

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are n eigenvalues (counted with their algebraic multiplicities), then $p(\lambda)$ can be expressed as

$$\det(\lambda I_n - A) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n).$$

Consequently, if $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct eigenvalues, then

$$\det(\lambda I_n - A) = (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k},$$

where $m_1, m_2, \ldots, m_k$ are positive integers satisfying $m_1 + m_2 + \cdots + m_k = n$.

Determinant and trace in terms of eigenvalues

- i $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$.
- ii $\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.

Remark

By the Theorem, the expansion of $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$ has form

$$p(\lambda) = \lambda^n - \mathrm{tr}(A)\lambda^{n-1} + \cdots + (-1)^n \det(A).$$

It implies that

- i $\det(A)$ is the constant term in $p(\lambda)$ multiplied by $(-1)^n$.
- ii $\mathrm{tr}(A)$ is the negative of the coefficient of λ^{n-1}.

Eigenvalues of a 2×2 matrix

Consider a general 2×2 matrix with real entries $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The characteristic polynomial of A is

$$p(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = \lambda^2 - (a+d)\lambda + (ad - bc).$$

Using the formulas $\mathrm{tr}(A) = a + d$ and $\det(A) = ad - bc$ to write p as

$$p(\lambda) = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A).$$

Since $p(\lambda)$ is a quadratic polynomial with real coefficients, we have

Number of eigenvalues

1. A has two distinct real eigenvalues if $\mathrm{tr}(A)^2 - 4\det(A) > 0$.
2. A has one real eigenvalue of multiplicity 2 if $\mathrm{tr}(A)^2 - 4\det(A) = 0$.
3. A has two complex conjugate eigenvalues if $\mathrm{tr}(A)^2 - 4\det(A) < 0$.

Eigenvalues of 2×2 symmetric matrix

If $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ is a symmetric matrix with real entries, then A has real eigenvalues. Moreover, A has one repeated eigenvalue if and only if $a = d$ and $b = 0$, in which case the eigenvalue is $\lambda = a$.

Eigenspaces of 2×2 symmetric matrix

1. If A has one repeated eigenvalue, then the eigenspace is all of $\mathbb{R}^2$.
2. If A has two distinct real eigenvalues, then the eigenspaces are perpendicular lines through the origin of $\mathbb{R}^2$.