Department of Mathematics

SF1624/SF1684
Algebra and Geometry
Year 2018/2019

Problems for Seminar 6

Check the canvas page of the course for information on how seminars are run and what you are expected to do before and during the seminars.

The seminar starts with a test. The problem will be about diagonalising a square matrix.

In the seminar, the following problems will be discussed.

Problem 1. Let $T = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ be the base change matrix from the basis $\mathcal{V}$ to the basis $\mathcal{W}$ of a subspace U of $\mathbb{R}^4$.

(a) Find a base change matrix from the basis $\mathcal{W}$ to the basis $\mathcal{V}$.
(b) Let $f\colon U \to U$ be a linear map such that $[f]_{\mathcal{W}} = \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}$. Find $[f]_{\mathcal{V}}$.

(Here $[f]_{\mathcal{B}}$ denotes the matrix for the map f with respect to the basis $\mathcal{B}$.)

Problem 2. Consider the following map:

$$F : \mathbb{R}^2 \to \mathbb{R}^2, \quad F(x,y) = (0,x)$$

(a) Find all eigenvalues and corresponding eigenspaces of F.
(b) Determine if the matrix for F is diagonalizable.

Problem 3. Let

$$A = \begin{bmatrix} 3 & a & 0 \\ 0 & 4 & 1 \\ 0 & 2 & 5 \end{bmatrix},$$

where a is a real parameter.

(a) Find the eigenvalues of A and eigenspaces corresponding to each eigenvalue.
(b) For which a is A diagonalisable?
(c) For $a = 0$, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Problem 4. We are given the matrix

$$A = \begin{bmatrix} 1 & 6 \\ 3 & -2 \end{bmatrix}.$$

(a) Find all eigenvalues and corresponding eigenvectors for A.
(b) Find a matrix U and a diagonal matrix D such that $A = UDU^{-1}$.
(c) Compute $A^{123} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Problem 5. The quadratic form Q on $\mathbb{R}^2$ is given by

$$Q(\vec{x}) = x_1^2 + x_1x_2 + x_2^2.$$

(a) Determine the symmetric matrix A which satisfies $Q(\vec{x}) = \vec{x}^T A\vec{x}$.
(b) Determine whether Q is positive definite, negative definite, positive semidefinite, negative semidefinite, or indefinite.

Problem 6. Which of the following sets are vektor spaces? Find a basis and the dimension for those that are.

(a) All vectors $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ i $\mathbf{R}^4$ such that $x + y + z - w = 1$
(b) All polynomial functions $f \colon \mathbb{R} \to \mathbb{R}$ of degree ≤ 5 (i.e. $f(x) = a + bx + cx^2 + dx^3 + fx^4 + gx^5$)
(c) All invertible 3×3-matrices
(d) All 3×3-matrices that satisfy $A^T = -A$. Here, A^T denotes the transpose matrix of A.

Miscellaneous

Here are some other topics that are important and interesting to discuss.

- What is the relationship between symmetric and orthogonal matrices?
- Why are symmetric matrices diagonalizable?
- What is a quadratic form and how does one classify them?