Department of Mathematics

SF1624/SF1684
Algebra and Geometry
Year 2018/2019

Problems for Seminar 5

Check the canvas page of the course for information on how seminars are run and what you are expected to do before and during the seminars.

The seminar starts with a test. The problem will be about determining an orthonormal basis for a given vector subspace of $\mathbb{R}^n$.

In the seminar, the following problems will be discussed.

Problem 1. Consider the matrix

$$A = \begin{bmatrix} -1 & -1 & 0 & 1 \\ -3 & -1 & 1 & 1 \\ 2 & 0 & -1 & 0 \end{bmatrix}.$$

(a) Determine a basis for the nullspace, $\mathrm{Null}(A)$.
(b) Determine a basis for the column space, $\mathrm{Col}(A)$.

Problem 2. Let

$$A = \begin{bmatrix} -2 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(a) Determine all vectors which lie in both Col(A) and Col(B). Explain why all vectors which lie in both Col(A) and Col(B) form a subspace of $\mathbb{R}^3$, and compute its dimension.
(b) Give a vector in Col(A) which does not lie in Col(B).

Problem 3. Let

$$\mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

be the standard basis for $\mathbb{R}^2$, and let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}.$$

(a) Show that $\mathcal{B}$ is a basis for $\mathbb{R}^2$.
(b) Compute the coordinate vector $[\vec{v}]_{\mathcal{B}}$ for the vector $\vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

(c) Compute matrices M and N such that

$$[\vec{x}]_{\mathcal{E}} = M\,[\vec{x}]_{\mathcal{B}} \quad \text{och} \quad [\vec{x}]_{\mathcal{B}} = N\,[\vec{x}]_{\mathcal{E}}$$

for all vectors $\vec{x}$ i $\mathbb{R}^2$.

Problem 4. A line $y = kx + m$ is to fit the points $(-2, 1)$, $(1, 2)$, $(4, 2)$, and $(7, 6)$.

(a) Determine the values of the constants k and m giving the bet fit in the sense of least squares.

(b) Sketch the line together with the points in a coordinate system and illustrate what it is that is minimized for these values of the constants.

Problem 5. Let $L\colon \mathbb{R}^2 \to \mathbb{R}^3$ an arbitrary but unspecified map.

(a) Why is the dimension of the image $\mathrm{Im}(T)$ of T at most 2?

(b) Let $\vec{b}$ be a vector in $\mathbb{R}^3$ which lies outside of the range $\mathrm{Im}(T)$. Explain how one can find the vectors $\vec{x}$ minimizing $\|L(\vec{x}) - \vec{b}\|$.

(c) Apply b) to find the smallest value of $\|L(\vec{x}) - \vec{b}\|$, where $L(\vec{x}) = (x_1, -x_2, x_1 + x_2)$, and $\vec{b} = (1, 2, 3)$.

Miscellaneous

Here are some other topics that are important and interesting to discuss.

- What is an orthogonal matrix?
- What is the purpose of the least squares method?
- What is the connection between the least squares method and (orthogonal) projections?