

## Homework 1

Information about homework:

- If correct solutions to questions 2-6 are handed in before the homework 1 deadline, 1 bonus point will be awarded for the final exam. The deadline is visible in CANVAS. If solutions to questions 2-6 handed in before that date are not correct, they have to be redone, but the second time without yielding bonus points for the exam. Solutions must be clearly written, and easy to follow. If not, they will not generate bonus points, and must be redone. It is highly recommended that the answers are done on a computer (for instance LaTeX or word). Task 2-4,6-8 can be done in groups of two (single person groups are allowed). Please submit, not later than midnight of the date of the homework 1 deadline:

- written solutions (in the form of a PDF-file)
- your matlab (or julia) programs for the problems

The files should be uploaded in CANVAS → SF2524 → Assignments → Homework 1

- If task 1 is solved before the *wiki-deadline*, 0.5 wiki-bonus will be awarded. If you collect  $y$  wiki-bonus points during the course, the grade limits for grade A and B will be reduced by  $y$  points. You submit your wiki tasks by specifying questions numbers (before wiki deadline) in

CANVAS → SF2524 → Assignments → Wiki part homework 1

1. Create problems and solutions on the course training wiki. This task is optional but can give wiki-bonus. In the wiki, create

- in Block 1-2:  $x$  exercise problems (per person), without solutions.
- in Block 1-2:  $x$  solutions (per person) to problems which do not yet have a solution. Don't do the problems you created.

If you are attending SF2524  $x \geq 2$ . If you are attending SF3580  $x \geq 3$ . If you are attending SF3580, but have completed SF2524 in your master studies  $x \geq 4$ . Moreover, SF3580 students have to *pose* at least 1 problem three days before the deadline. In order to get

Course training wiki: [http://gragg.math.kth.se/sf2524/merge\\_group\\_pages2.php?name=54162](http://gragg.math.kth.se/sf2524/merge_group_pages2.php?name=54162)

QR-code link:



good problem ideas, you might want to do the wiki-problems in parallel or after solving the problems below.

2. Consider the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & 9 \end{bmatrix}.$$

Plot the eigenvalue error as a function of iterate for the following algorithms in a *semilog* plot, and relate with the convergence theory.

- Power method with the starting vector  $x_0^T = (1, 1, 1)$ .
- Rayleigh quotient iteration with the starting vector  $x_0^T = (1, 1, 1)$ .
- Change the entry of the matrix in element row 1, column 3, to  $a_{1,3} = 4$  and again run the Rayleigh quotient iteration with  $x_0^T = (1, 1, 1)$ . Explain why we obtain slower convergence in comparison to (b).

3. In this exercise we will investigate the performance of different versions of the Gram-Schmidt orthogonalization when it is combined with the Arnoldi method. Consider the matrix  $A$  constructed with the following command

```
rand('seed',0); A=gallery('wathen',nn,nn); (1)
```

- Modify the orthogonalization in `arnoldi.m` available from the course we page. Apply it to (1) and generate the values in the following table, where `time` is the CPU-time and `orth` =  $\|Q_m^T Q_m - I\|$  is an indicator of the orthogonality of the basis, and  $m$  the number of iterations in Arnoldi's method.

	single GS		modified GS		double GS		triple GS	
	time	orth	time	orth	time	orth	time	orth
$m = 5$								
$m = 10$								
$m = 20$								
$m = 50$								
$m = 100$								

Select  $nn \geq 10$  such that the maximum computation time for  $m = 100$  is approximately 10 minutes on your computer.

- Interpret the result in (a). Is one version "best" in this setting? If so, in what sense is it "best"? Discuss the fairness of the comparison.

4. We shall here investigate a primitive variant of the Arnoldi method. Let  $K_m$  be a matrix with the iterates of the power method, that is, let  $K_m := [b, Ab/\|Ab\|, \dots, A^{m-1}b/\|A^{m-1}b\|] \in \mathbb{R}^{n \times m}$ .

You may in this exercise use `eig(A)` as a reference/exact solution.

Use tic-toc to measure performance in MATLAB. In order to get reliable estimates, you may need the tricks described here: [http://se.mathworks.com/help/matlab/matlab\\_prog/measure-performance-of-your-program.html](http://se.mathworks.com/help/matlab/matlab_prog/measure-performance-of-your-program.html).

Background Galerkin method: The Galerkin method is a fundamental technique in the finite element method for partial differential equations. The approximation is generated from the bilinear form  $a(\cdot, \cdot)$  and  $f(\cdot)$  on vector space  $V = \text{span}(v_1, \dots, v_m)$  and is defined by the equality

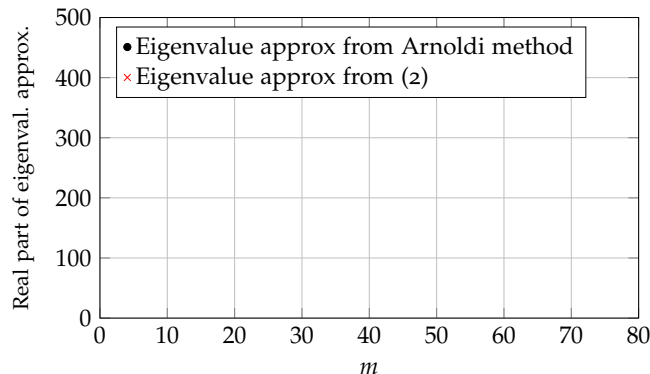
$$\sum_{i=1}^m w_i a(v_i, v_j) = f(v_j)$$

- (a) The approximation stemming from Galerkin method applied to the bilinear form associated with the eigenvalue problem ( $a(u, v) = u^T A v - \mu u^T v, f(v) = 0$ ) for the subspace spanned by the columns of  $K_m$  satisfies

$$\mu K_m^T K_m w = K_m^T A K_m w \quad (2)$$

You may assume that  $\mathcal{K}_m(A, b)$  has dimension  $m$ . Prove that the approximation (2) is identical to the approximation generated by Arnoldi's method for eigenvalue problem. No prior knowledge about the Galerkin method is required to solve the problem.

- (b) By solving (2) directly with eig, we have a method which requires only computation of  $K_m^T K_m$  and  $K_m^T A K_m$ , and does not require any orthogonalization. Apply this method to (1) and compare with the Arnoldi method. Use double GS as orthogonalization. Generate the following figure for  $m = 1, 2, \dots$ : Use a random starting vector, with the same starting vector for all simulations.



Hint: Recall that if  $V_m$  is an orthogonal basis of  $\mathcal{K}_m(A, b)$ , then the Arnoldi approximation is eigenvalues of  $V_m^T A V_m$ . Maybe there exists a non-singular matrix  $R$  such that  $K_m = V_m R$ ? Show that  $R^T R = K_m^T K_m$  by considering  $V_m^T V_m$ .

Note that  $K_m^T A K_m$  has  $m$  eigenvalues, so the plot will consist of several curves/points.

- (c) Interpret the result of (a)-(b). What do we expect in exact arithmetic? What do we observe? Which approach is better? Why?

5. Do the quiz "Lanczos algorithm" on CANVAS. You can do the quiz as many times as you like. This task is individual and should not be done as a group. No reply in the PDF-file is required for this problem.

6. Download and load the matrix

<http://www.math.kth.se/na/SF2524/matber14/Bwedge.mat>

- (a) The eigenvalues are given in the variable B\_eigvals. Plot the eigenvalues and mark the eigenvalues you will have the fastest convergence with Arnoldi's method.
- (b) In the same figures as in (a), indicate circles and estimate a convergence factor. The convergence factor is here meant  $\alpha < 1$  such that the error  $\sim \alpha^m$ .
- (c) Generate figures with the Ritz values for the Arnoldi method, with double GS for  $m = 2, 4, 8, 10, 20, 30, 40$ . Clearly indicate

MATLAB users: Use the command `load('Bwedge.mat')` to access the matrix  $B$  and eigenvalues.

Julia users: see file instructions  
CANVAS  $\rightarrow$  SF2524  $\rightarrow$  Modules  $\rightarrow$  Julia instructions

which eigenvalue you expect fast convergence based on (a). How many iterations are needed in order to get an eigenvalue error approximately  $10^{-10}$

Modify your algorithm such that it carries out shift-and-invert Arnoldi method with shift  $\sigma = -11 + 2i$ .

- (d) Plot the eigenvalues of the transformed matrix  $B = (A - \sigma I)^{-1}$  and indicate as in (a)
- (e) How many iterations are needed to achieve eigenvalue accuracy  $10^{-10}$ . Is it faster or slower than (b)? Does the theory predict faster or slower than (c)? You can justify your reasoning with figures.

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Only for PhD students taking the course SF3580 *Numerical linear algebra*. You can choose to do either problem 7 or problem 8.

- 7. Exercise about restarting. Use the matrix Bwedge.mat in the simulations.

- (a) Implement an explicit restarting strategy for the Arnoldi method as follows. Take a linear combination (with unit weights) of the Ritz vectors generated by  $m$  iterations of the Arnoldi method as a new starting vector. Use the Ritz vectors corresponding to the  $k$  largest Ritz values. Extract the Ritz vectors as follows:

```
[Q,H]=arnoldi(A,b,m);
[V,D]=eig(H(1:end-1,1:end)); d=diag(D);
[Y,I]=sort(-abs(d));
ritz_vals=d(I(1:k));
ritz_vecs=Q(:,1:end-1)*V(:,I(1:k));
```

Carry out the experiments for  $m = 10, k = 5$  and  $m = 20, k = 10$  with 100 restarts. Plot the eigenvalue approximation as a function of restart. Does the restart strategy work in practice?

- (b) Use the reference code `arnupd.m` and `arnoldi_sorensen.m` which are implementations of an implicit restarting strategy. Run the algorithm with same restarting parameters as in (a) and explain the difference. Implicit restarting is in general to prefer over explicit restarting. Why?
- 8. SVD: Singular value decomposition. If you have not heard of SVD, you may need to look it up, in particular the Eckart-Young theorem.
- (a) What are the relationship between the singular values of the matrix  $A \in \mathbb{R}^{n \times m}$  and the eigenvalues and eigenvectors of

$$B = \begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}.$$

The programs `arnoldi_sorensen.m` and `arnupd.m` are reference implementations of *Implicit application of polynomial filters in a k-step arnoldi method*, D. C. Sorensen, *SIAM J. Matrix Anal. Appl.* Vol. 13, No. 1, pp. 357-385, 1992. It is not necessary to read the details of the paper to solve the homework. The programs are available from <http://www.math.kth.se/~eliasj/NLA/arnupd.m> Another good reference for implicit restarting are provided in the lecture notes: <http://people.inf.ethz.ch/arbenz/ewp/Lnotes/chapter10.pdf>



- (b) Download the movie `india_driving1` from <https://kth.instructure.com/groups/54162/files>. Reshape the movie to a matrix. Compute the SVD (with `svd(A)`) and construct the best rank one approximation

$$\tilde{A} = u_1 s_1 v_1^T$$

According to Eckart-Young theorem, this corresponds to the largest singular value and corresponding vector. Visualize and interpret the first column of  $\tilde{A}$  as an image. Explain (briefly) why this image looks this way.

See CANVAS “Julia instructions” for how to load and manipulate videos frames in Julia.

- (c) Download the larger movie file `market`. Reshape the movie to a matrix. The `svd` does not work on this matrix (on most computers) since the matrix is too large and the `svd`-algorithm runs out of memory. Use (a) and the Lanczos method to compute the singular vector and solve visualize as in (b).