

o Fundamental concepts

This is a short summary of certain important fundamental techniques, which are not a part of the course but will be used several times.

o.1 Orthogonal matrices and QR factorization

The QR-factorization is a factorization involving an orthogonal matrix Q and an upper triangular matrix R .

Definition 0.1.1 (Orthogonal matrix). $Q \in \mathbb{R}^{n \times m}$ is called an orthogonal matrix if

$$Q^T Q = I.$$

For complex matrices, the corresponding property is called unitary: $Q^* Q = I$.

Properties:

- (i) The columns of an orthogonal matrix are orthonormal.
- (ii) If $n = m$, then $Q^T = Q^{-1}$.
- (iii) If $n = m$, then $Q Q^T = I$.

Note that (iii) is not satisfied if Q is a rectangular matrix ($n \neq m$). For instance

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is an orthogonal matrix since $Q^T Q = I$, but

$$Q Q^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Theorem 0.1.2 (Uniqueness of QR-factorization). For any matrix $A \in \mathbb{R}^{n \times m}$, $n \leq m$ there exists an orthogonal matrix $Q \in \mathbb{R}^{n \times n}$ and an upper triangular matrix $R \in \mathbb{R}^{m \times m}$ such that

$$A = QR.$$

Orthogonal matrices are important in matrix computations, most importantly when the matrix represents a basis of a vector space. Many properties of the vector space are not robust with respect to rounding errors if the basis is not orthogonal.

The identity matrix will be denoted $I \in \mathbb{R}^{k \times k}$, where k is given by the context we use it.

A QR-factorization can be computed with the matlab command $[Q,R]=qr(A)$. It will however in general not return the solution with positive diagonal elements.

Moreover, if A is non-singular, the diagonal elements of R can be chosen positive, and the decomposition where the diagonal elements of R are positive is unique.

The QR-decomposition is the underlying method to solve overdetermined linear systems of equations, for instance with the backslash operator in matlab when the matrices are rectangular. The decomposition can be computed in a finite number of operations with for instance Householder reflectors or Givens rotations, which will be used in this course.

0.2 Jordan canonical form (JCF)

The Jordan canonical form, also sometimes the Jordan form or the Jordan decomposition, is a transformation that brings the matrix to a certain block diagonal form.

Definition 0.2.1 (Jordan canonical form). *The Jordan decomposition of a matrix $A \in \mathbb{R}^{n \times n}$ is an invertible matrix $X \in \mathbb{R}^{n \times n}$ and a block diagonal matrix*

$$D = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix}$$

where

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \in \mathbb{R}^{n_i \times n_i}$$

such that

$$A = XDX^{-1}.$$

The matrix J_i is called a Jordan block, or sometimes a Jordan matrix.

Example of Jordan decomposition

The Jordan decomposition of

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

is represented by

$$D = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

since $A = XDX^{-1}$. Note that the eigenvalue $\lambda = 3$ has two Jordan blocks, one of size $n_1 = 2$ and one of size $n_2 = 1$.

The Jordan canonical form is typically treated in basic linear algebra courses, but also used in the study of stability theory in for instance systems theory and differential equations.

The Jordan decomposition is a *eigenvalue revealing* decomposition, since the eigenvalues of A are the diagonal of the J -matrix.



Theorem 0.2.2 (Existence and uniqueness of Jordan decomposition). All matrices $A \in \mathbb{R}^{n \times n}$ have a Jordan decomposition. The (unordered) set of Jordan blocks is unique for any given matrix.

Definition 0.2.3. If a matrix A has a Jordan decomposition where all Jordan blocks have size one (such that D is a diagonal matrix) the matrix A is called a diagonalizable matrix.

Lemma 0.2.4. Suppose A is symmetric. Then A is diagonalizable and the X matrix in the Jordan decomposition can be chosen as an orthogonal matrix.

We say that the eigenvalues of a matrix A denoted $\lambda_1, \dots, \lambda_n$ are distinct if they are different from each other, that is

$$\lambda_i \neq \lambda_j \text{ when } i \neq j$$

This is a sufficient condition for diagonalizability.

Lemma 0.2.5. If $A \in \mathbb{R}^{n \times n}$ has n distinct eigenvalues, then A is diagonalizable.

0.3 Linear least squares problems

Let $c \in \mathbb{R}^n$ and let $B \in \mathbb{R}^{n \times m}$ be a tall skinny matrix ($n > m$), where the columns of B are linearly independent. The problem to determine a vector $z \in \mathbb{R}^m$ which minimizes the problem

$$\min_{z \in \mathbb{R}^m} \|Bz - c\|_2 \quad (*)$$

is usually referred to as the linear least squares problem. We will commonly use the corresponding minimizer

$$z_* = \operatorname{argmin}_{z \in \mathbb{R}^m} \|Bz - c\|_2 \Rightarrow \min_{z \in \mathbb{R}^m} \|Bz - c\|_2 = \|Bz_* - c\|_2.$$

The problem $(*)$ is sometimes denoted

$$Bz \approx c.$$

The solution to linear least squares problem are given by the normal equations

$$B^T Bz = B^T c, \quad (**)$$

which is a linear system of equation which has a unique solution.

Although $(**)$ gives a direct procedure how to compute a solution to $(*)$ by solving a linear system of equations, it is not the best way to compute the solution numerically. In MATLAB the backslash operator can compute solutions to linear least squares problems $z = B \backslash c$

The method behind backslash for tall and skinny matrices is based on computing a QR-factorization.