Department of Mathematics

SF1624
Algebra and Geometry
Year 2017/2018

Problems for Seminar 5

Check the canvas page of the course for information on how seminars are run and what you are expected to do before and during the seminars.

Start by doing the simpler recommended exercises from the book:

5.1: A3
5.2: A2
5.4: A1
6.1: A1
6.2: A1

The seminar starts with a test. The problem will be about computing the eigenvalues and eigenvectors of a matrix.

In the seminar, the following problems will be discussed.

Problem 1. Explain, using the properties of determinants, why quadratic matrices A satisfy $\det(A^TA) = (\det(A))^2$. Apply this to compute $\det(A^TA)$ in the special case where

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 0 & 3 & 1 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}.$$

Problem 2. Let

$$A = \begin{bmatrix} -11 & 9 & 6 \\ -8 & 6 & 2 \\ -6 & 6 & 7 \end{bmatrix}.$$

(a) Show, without computing the characteristic polynomial, that $\lambda = 1$ is an eigenvalue of A.
(b) Find all eigenvalues and corresponding eigenvectors of A.
(c) Find a diagonal matrix D and a matrix P such that $A = PDP^{-1}$.

Problem 3. Let
$$A = \begin{bmatrix} 2 & 2 & 2 \\ -\frac{5}{2} & 0 & -2 \\ 0 & -2 & 0 \end{bmatrix}$$

(a) Determine all (real) eigenvalues and corresponding eigenvectors of A. *Tip:* one of the eigenvalues is $\lambda = 2$.
(b) Determine all eigenvalues and corresponding eigenvectors of A^2.
(c) Find a diagonal matrix D and a matrix P such that $A^2 = PDP^{-1}$.
(d) Use this to compute $A^{15} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Problem 4. Let L be the line given by the equation $4x - 3y = 0$. Let $S\colon \mathbf{R}^2 \to \mathbf{R}^2$ be reflection through that line, and let $P\colon \mathbf{R}^2 \to \mathbf{R}^2$ be projection onto that line.

(a) Determine, geometrically without doing any computations, wheter there exist non-zero vectors $\vec{x}$ such that
$$a) \quad S(\vec{x}) = 1 \cdot \vec{x} \quad b) \quad P(\vec{x}) = 1 \cdot \vec{x} \quad c) \quad S(\vec{x}) = -1 \cdot \vec{x} \quad d) \quad P(\vec{x}) = 0 \cdot \vec{x}.$$
(b) Show that if you have a non-zero vector $\vec{x}$ satisfying a), b), c) or d), then any point on the line passing through origin and $\vec{x}$, will also satisfy a), b), c) or d), respectively.
(c) Determine the standard matrices for S and P, and determine their characteristic polynomials: If A is a 2×2-matrix, then its characteristic polynomial is $\det(\lambda \cdot I - A)$, where λ is a variable, and I is the identity matrix.
(d) Determine the roots of the the characteristic polynomials you found above.
(e) For each root λ_0 of the characteristic polynomial of the matrix representing S, determine all vectors $\vec{x}$ such that $S(\vec{x}) = \lambda_0 \vec{x}$.
(f) For each root λ_0 of the characteristic polynomial of the matrix representing P, determine all vectors $\vec{x}$ such that $P(\vec{x}) = \lambda_0 \vec{x}$.

MISCELLANEOUS

Here are some other topics that are important and interesting to discuss.

- What are different ways to compute determinants? Discuss advantages and disadvantages.
- What is the geometric significance of eigenvectors of a linear map? For instance for the eigenvalues 0 or 1?
- Why are eigenvectors for different eigenvalues linearly independent?
- Is an $n \times n$-matrix diagonalizable if it has n different eigenvalues? Is that a necessary condition?