Department of Mathematics

SF1624
Algebra and Geometry
Year 2017/2018

Problems for Seminar 4

Check the canvas page of the course for information on how seminars are run and what you are expected to do before and during the seminars.

Start by doing the simpler recommended exercises from the book:

3.1: A2, A7
3.2: A1, A6
3.4: A1, A2, A3, A6
4.5: A3
4.6: A1, A5

The seminar starts with a test. The problem will be about computing the standard matrix of a geometrically defined map and finding a basis of its nullspace of column space.

In the seminar, the following problems will be discussed.

Problem 1. The map $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ is given by the matrix

$$A = \begin{bmatrix} -1 & 2 & 1 \\ 2 & -4 & -2 \end{bmatrix}.$$

(a) Find two vectors $\vec{u}$ and $\vec{v}$ that constitute a basis for the kernel, $\ker(T)$.
(b) Determine an equation for the plane that is $\ker(T)$.

Problem 2. This exercise is about linear maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ and their standard matrices.

(a) Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be the anticlockwise rotation around the origin by an angle of $90°$ ($\pi/2$ radians). Determine the standard matrix A for T_1.
(b) Let $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection about the line $y = -x$. Determine the standard matrix B for T_2.
(c) Determine the standard matrix C for the composition $T_2 \circ T_1$.
(d) The map $T_2 \circ T_1$ is a reflection. About which line? Explain your answer.

Problem 3. The linear map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfies

$$T\left(\begin{bmatrix} 5 \\ 10 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 11 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 0 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

(a) Determine the matrix A associated to T.

(b) Verify that A is its own inverse.

Problem 4. The linear map $R\colon \mathbf{R}^3 \to \mathbf{R}^3$ is the rotation around the vector $\begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$ with angle $\frac{2}{3}\pi$ according to the right-hand rule. A basis $\mathcal{B}$ of $\mathbf{R}^3$ is given by

$$\mathcal{B} = \left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix} \right\}.$$

(a) Compute the standard matrix for R. *Tip:* Observe that (but explain why) the standard basis vectors are mapped to one another.
(b) Compute the matrix for R with respect to the basis $\mathcal{B}$.
(c) The linear map $L\colon \mathbf{R}^3 \to \mathbf{R}^3$ is given by $L(\vec{x}) = R(\vec{x}) - x$. Compute the rank of L.
(d) Find a basis for the null space (kernel) of L.

Problem 5. A map f is sought that satisfies

$$f\left(\begin{bmatrix}1\\2\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad f\left(\begin{bmatrix}2\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad f\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\\1\end{bmatrix}, \quad f\left(\begin{bmatrix}1\\-1\\-2\end{bmatrix}\right) = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

(a) Why is there no such map that is linear?
(b) How must we change the last value, $\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$, such that such a linear map f exists?
(c) Find the standard matrix for f (with the last value corrected as in b).

Problem 6. Let $\vec{v}_1 = \begin{bmatrix}1\\2\end{bmatrix} \in \mathbf{R}^2$, and define additional vectors $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$ by rotating $\vec{v}_1$ anticlockwise by $45°$, $90°$, and $135°$, respectively. The linear map $S\colon \mathbf{R}^2 \to \mathbf{R}^2$ is the reflection at the line spanned by $\vec{v}_1$.

(a) Find the matrix A for S with respect to the basis $\{\vec{v}_1, \vec{v}_3\}$.
(b) Find the matrix B till S with respect to the basis $\{\vec{v}_2, \vec{v}_4\}$.
(c) Find the base change matrix P from the basis $\{v_1, \vec{v}_3\}$ to the basis $\{\vec{v}_2, \vec{v}_4\}$.
(d) Compute $P^{-1}BP$ and explain why this yields A.

MISCELLANEOUS

Here are some other topics that are important and interesting to discuss.

- What is a linear map? Is every linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ a matrix map? How many rows and columns does the matrix of the map have?
- What is the nullspace/kernel and the image of a map? Which vectors constitute the nullspace and column space of a rotation and of a reflection?
- Does every matrix have an inverse? If not, how does one determine whether a matrix has an inverse, and how does one compute the inverse?