Products in cohomology

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Tensor product of complexes

Definition

Let $A^\bullet, B^\bullet$ be (co)chain complexes. We define their tensor product $(A \otimes B)^\bullet$ as follows. In degree n,

$$(A \otimes B)^n = \bigoplus_{i+j=n} A^i \otimes B^j.$$

The differential is defined as follows. Let $a^m \in A^m, b^n \in B^n$. Then

$$d(a^m \otimes b^n) = d_A(a^m) \otimes b^n + (-1)^m a^m \otimes d_B(b^n).$$

Similarly we define the tensor product of graded abelian groups, such as $H^*(X) \otimes H^*(Y)$.

Algebraic Kunneth theorem

Theorem

Suppose $A^\bullet, B^\bullet$ are chain complexes of free R-modules. There is a natural homomorphism:

$$H^*(A) \otimes_R H^*(B) \to H^*(A \otimes B).$$

This homomorphism is an isomorphism if $A^\bullet$ is a chain complex of free R-modules, and $H^n(A)$ is also a free R-module for every n.

Topology

Let X, Y be topological spaces. Let $\mathcal{U}, \mathcal{V}$ be open covers of X and Y respectively. Let $\mathcal{U} \times \mathcal{V}$ be the open cover of $X \times Y$ consisting of all products of elements of $\mathcal{U}$ and $\mathcal{V}$. Notice that there is a homomorphism

$$\begin{array}{ccc} C^0_{\mathcal{U}}(X; A) \otimes C^0_{\mathcal{V}}(Y; B) & \to & C^0_{\mathcal{U} \times \mathcal{V}}(X \times Y; A \otimes B) \\ \prod_{U \in \mathcal{U}} A^U \otimes \prod_{V \in \mathcal{V}} B^V & \to & \prod_{\mathcal{U} \times \mathcal{V}} (A \otimes B)^{U \times V} \\ (f\colon U \to A) \otimes (g\colon V \to B) & \to & f \otimes g\colon U \times V \to A \otimes B \end{array}$$

It turns out, that this homomorphism can be extended to a map of chain complexes

Theorem

There exists a natural homomorphism - in fact a chain homotopy equivalence

$$C^*_{\mathcal{U}}(X, A) \otimes C^*_{\mathcal{V}}(Y, B) \to C^*_{\mathcal{U} \times \mathcal{V}}(X \times Y; A \otimes B).$$

The Kunneth formula

The chain homomorphism induces a natural homomorphism

$$\check{\mathrm{H}}^*(X, A) \otimes \check{\mathrm{H}}^*(Y, B) \to \check{\mathrm{H}}^*(X \times Y; A \otimes B).$$

In particular, if $A = B = R$ is a ring, one gets a homomorphism

$$\check{\mathrm{H}}^*(X, R) \otimes_R \check{\mathrm{H}}^*(Y, R) \to \check{\mathrm{H}}^*(X \times Y; R).$$

Theorem

This homomorphism is an isomorphism if, for example, X is a CW complex with finitely many cells in each dimension, and $\check{\mathrm{H}}^*(X, R)$ is a free R-module (or more generally, flat).

Internal ring structure on cohomology

Suppose X is a space and R is a commutative ring. Then we have natural homomorphisms

$$\check{\mathrm{H}}^*(X, R) \otimes_R \check{\mathrm{H}}^*(X, R) \to \check{\mathrm{H}}^*(X \times X; R) \to \check{\mathrm{H}}^*(X, R).$$

This endows $\check{\mathrm{H}}^*(X, R)$ with the structure of a *graded commutative ring*. For whatever historical reasons, the multiplication in cohomology is denoted by the symbol $\cup$, and is called "cup product".

Example

$$\begin{aligned}
\check{\mathrm{H}}^*(S^m \times S^n, \mathbb{Z}) &\cong \mathbb{Z}[u_m, u_n]/(u_m^2, u_n^2) \\
\check{\mathrm{H}}^*(\mathbb{C}P^n, \mathbb{Z}) &\cong \mathbb{Z}[u_2]/(u_2^{n+1}). \\
\check{\mathrm{H}}^*(\mathbb{R}P^n, \mathbb{Z}/2) &\cong \mathbb{Z}/2[x]/(x^{n+1}).
\end{aligned}$$

Using Gysin to calculate $H^*(\mathbb{C}P^n)$ and $H^*(\mathbb{R}P^n; \mathbb{Z}/2)$

The space $\mathbb{C}P^n$ fits in a principal $U(1)$-fibration

$$S^1 \to S^{2n+1} \to \mathbb{C}P^n.$$

Notice that it is a sphere bundle. In fact, one can identify S^{2n+1} with the sphere bundle of the vector bundle

$$S^{2n+1} \times_{U(1)} \mathbb{C} \to S^{2n+1}/_{U(1)} = \mathbb{C}P^n.$$

This is an orientable bundle, because $U(1) \subset SO(2)$. Therefore it has an Euler class $u \in H^2(\mathbb{C}P^n)$. The Gysin sequence has the following form

$$\cdots \to H^{i-2}(\mathbb{C}P^n) \xrightarrow{u} H^i(\mathbb{C}P^n) \to H^i(S^{2n+1}) \to H^{i-1}(\mathbb{C}P^n) \to \cdots$$

It follows that multiplication by u induces isomorphisms

$$H^0(\mathbb{C}P^n) \xrightarrow{\cong} H^2(\mathbb{C}P^n) \xrightarrow{\cong} \cdots \xrightarrow{H}{}^{2n}(\mathbb{C}P^n).$$

And therefore $H^*(\mathbb{C}P^n) \cong \mathbb{Z}[u]/(u^{n+1})$.

Applications: The Borsuk-Ulam theorem, non-existance of division algebra structure on $\mathbb{R}^n$ for $n \neq 2^k$.