Cech and sheaf cohomology

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An example of Cech cohomology

As usual, let $I = [0,1]$. Let us calculate $\check{\mathrm{H}}^*(I, A)$. Consider the cover $\mathcal{U} = \{[0, \frac{3}{4}), (\frac{1}{2}, 1]\}$. The groups $\check{\mathrm{H}}^*_{\mathcal{U}}(I; A)$ are calculated by the following complex:

$$\begin{array}{ccccccc}
n & 0 & & 1 & & 2 & \cdots \\
C^n_{\mathcal{U}} & A \oplus A & \xrightarrow{[1,-1]} & A & \to & 0 & \cdots \\
\check{\mathrm{H}}^n_{\mathcal{U}} & A & & 0 & & 0 & \cdots
\end{array}$$

Now let $\mathcal{U} = \{[0, b_1), (a_2, b_2), \ldots, (a_n, 1]\}$, where for each k, $a_k < b_{k-1} < a_{k+1}$. Then the Cech complex has the following form

$$\begin{array}{ccccccc}
n & 0 & & 1 & & 2 & \cdots \\
C^n_{\mathcal{U}} & A^n & \longrightarrow & A^{n-1} & \to & 0 & \cdots \\
\check{\mathrm{H}}^n_{\mathcal{U}} & A & & 0 & & 0 & \cdots
\end{array}$$

Every open cover of I has a refinement of this type. From here one can easily reach the (unsurprising) conclusion:

$$\check{\mathrm{H}}^n(I; A) = \begin{cases} A & n = 0 \\ 0 & n > 0 \end{cases}$$

Homotopy invariance

"Similarly" to our calculation of $\check{\mathrm{H}}^n(I; A)$, one can prove the following

Lemma

Let X be a paracompact space. The canonical inclusions $i_0, i_1 \colon X \hookrightarrow X \times I$ induce **isomorphisms** *(which are necessarily the same)*

$$i_0^*, i_1^* \colon \check{\mathrm{H}}^*(X \times I; A) \xrightarrow{\cong} \check{\mathrm{H}}^*(X; A).$$

As a consequence we have the following important theorem

Theorem (Homotopy invariance)

Let $f, g \colon X \to Y$ be homotopic maps between paracompact spaces. Then they induce the same homomorphism

$$\check{\mathrm{H}}^*(Y; A) \to \check{\mathrm{H}}^*(X; A).$$

The Mayer-Vietoris sequence

Suppose $X = U_1 \cup U_2$, where U_i are open subsets. There is a LES

$$\begin{array}{ccccccccc} 0 & \to & \check{\mathrm{H}}^0(X;A) & \xrightarrow{(i_1^0,i_2^0)} & \check{\mathrm{H}}^0(U_1;A) \oplus \check{\mathrm{H}}^0(U_2;A) & \xrightarrow{(j_1^0-j_2^0)} & \check{\mathrm{H}}^0(U_1 \cap U_2;A) & \to \\ & \to & \check{\mathrm{H}}^1(X;A) & \xrightarrow{(i_1^1,i_2^1)} & \check{\mathrm{H}}^1(U_1;A) \oplus \check{\mathrm{H}}^1(U_2;A) & \xrightarrow{(j_1^1-j_2^1)} & \check{\mathrm{H}}^1(U_1 \cap U_2;A) & \cdots \end{array}$$

Proof: A denotes an abelian group, and also the sheaf represented by A: $O \mapsto A^O$. Let $A_1 = i_1^* i_{1!} A$ be the sheaf $O \mapsto A^{O \cap U}$. Def. A_2, A_{12} similarly. We get a sort of exact sequence of presheaves:

$$0 \to A \xrightarrow{(i_1,i_2)} A_1 \oplus A_2 \xrightarrow{j_1-j_2} A_{12} (\dashrightarrow 0).$$

This is a *left exact* sequence of presheaves and it is short exact when evaluated on a set O that is contained either in U_1 or in U_2. It follows that if $\mathcal{O}$ is an open cover of X, that is subordinate to $\{U_1, U_2\}$, there is a short exact sequence of Cech complexes

$$0 \to C^*_{\mathcal{O}}(X;A) \xrightarrow{(i_1^*,i_2^*)} C^*_{\mathcal{O}}(U_1;A) \oplus C^*_{\mathcal{O}}(U_2;A) \xrightarrow{j_1^*-j_2^*} C^*_{\mathcal{O}}(U_1 \cap U_2;A) \to 0.$$

From here we get a LES of cohomology groups $\check{\mathrm{H}}^*_{\mathcal{O}}(-;A)$. The poset of subordinate open covers is cofinal, so passing to colimits we obtain a LES of cech cohomology groups

Examples

Using homotopy invariance, the Mayer-Vietoris sequence, and induction, one can easily calculate the cohomology groups of a sphere:

$$\check{\mathrm{H}}^n(S^d;A) = \begin{cases} A & n = 0, \text{ or } d \\ 0 & \text{otherwise} \end{cases}$$

Remark

It is often convenient to consider reduced cohomology:

$$\widetilde{\mathrm{H}}^*(X;A) := \ker\left(\check{\mathrm{H}}^*(X;A) \to \check{\mathrm{H}}^*(*;A)\right).$$

Reduced cohomology is almost the same as unreduced cohomology, except that it knocks of a factor of A in degree zero. For example $\widetilde{\mathrm{H}}^*(*;A) = 0$, and $\widetilde{\mathrm{H}}^*(S^d;A)$ only has a copy of A in degree d.

The cohomology of *CW-complexes* is often tractable. Let X be a CW-complex. For each d, let X^d be the d-skeleton of X. Thus X is filtered by subspaces $X^{d-1} \subset X^d \subset \cdots$. Let C_n be the set of n-dimensional cells of X.

Lemma

For each d, there is a long exact sequence

$$\cdots \to \check{\mathrm{H}}^n(X^d; A) \to \check{\mathrm{H}}^n(X^{d-1}; A) \to \widetilde{\mathrm{H}}^n(S^{d-1}; A)^{C_d} \to \check{\mathrm{H}}^{n+1}(X^d; A) \cdots .$$

In fact, one can organize things even better by splicing the exact sequences. One obtains a chain complex of the following form

$$A^{C_0} \to A^{C_2} \to \cdots \to A^{C_d} \to \cdots$$

This is called the *cellular chain* complex of X. Its cohomology is isomorphic to the cech cohomology (or any other cohomology satisfying the Eilenberg-Steenrod axioms).

Example: projective spaces

The complex projective space $\mathbb{C}P^n$ is obtained from $\mathbb{C}P^{n-1}$ by attaching a cell of dimension $2n$. Thus $\mathbb{C}P^n$ has a cell structure with a single cell in each even dimension. It follows immediately that the cohomology of $\mathbb{C}P^n$ is given by the following formula:

$$\check{\mathrm{H}}^i(\mathbb{C}P^n; A) = \begin{cases} A & i = 2l \leq 2n \\ 0 & \text{otherwise} \end{cases}$$

The real projective space requires a little more work. The space $\mathbb{R}P^n$ is obtained by attaching a cell of dimension n to $\mathbb{R}P^{n-1}$. Thus $\mathbb{R}P^n$ has a cell structure with a single cell in each dimension. It follows that the cellular chain complex of $\mathbb{R}P^n$ has the following form

$$\begin{array}{ccccccccc} 0 & & 1 & & 2 & & \cdots & & n \\ A & \xrightarrow{d^0} & A & \xrightarrow{d^1} & A & \xrightarrow{d^2} & \cdots & \xrightarrow{d^{n-1}} & A \end{array}$$

Lemma

d^i is multiplication by 2 for odd i and is zero for even i.

products

The lemma about the cohomology of $X \times I$ is a special case of a general result about the cohomology of products. Before we state the result, let us review tensor products of complexes.

Definition

Let $A^\bullet, B^\bullet$ be (co)chain complexes. We define their tensor product $(A \otimes B)^\bullet$ as follows. In degree n,

$$(A \otimes B)^n = \bigoplus_{i+j=n} A^i \otimes B^j.$$

The differential is defined as follows. Let $a^m \in A^m, b^n \in B^n$. Then

$$d(a^m \otimes b^n) = d_A(a^m) \otimes b^n + (-1)^m a^m \otimes d_B(b^n).$$

Similarly we define the tensor product of graded abelian groups, such as $H^*(X) \otimes H^*(Y)$.

Algebraic Kunneth theorem

Theorem

Suppose $A^\bullet, B^\bullet$ are chain complexes of free R-modules. There is a natural homomorphism:

$$H^*(A) \otimes_R H^*(B) \to H^*(A \otimes B).$$

This homomorphism is an isomorphism if $A^\bullet$ is a chain complex of free R-modules, and $H^n(A)$ is also a free R-module for every n.

Topology

Let X, Y be topological spaces. Let $\mathcal{U}, \mathcal{V}$ be open covers of X and Y respectively. Let $\mathcal{U} \times \mathcal{V}$ be the open cover of $X \times Y$ consisting of all products of elements of $\mathcal{U}$ and $\mathcal{V}$. Notice that there is a homomorphism

$$C^0_{\mathcal{U}}(X; A) \otimes C^0_{\mathcal{V}}(Y; B) \to C^0_{\mathcal{U}\times\mathcal{V}}(X \times Y; A \otimes B)$$

The homomorphism sends a pair of functions $(f : U \to A) \otimes (g : V \to A)$ to the function $f \otimes g : U \times V \to A \otimes B$. It turns out, that this homomorphism can be extended to a map of chain complexes

Theorem

There exists a natural homomorphism - in fact a chain homotopy equivalence

$$C^*_{\mathcal{U}}(X, A) \otimes C^*_{\mathcal{V}}(Y, B) \to C^*_{\mathcal{U}\times\mathcal{V}}(X \times Y; A \otimes B).$$

The Kunneth formula

The chain homomorphism induces a natural homomorphism

$$\check{\mathrm{H}}^*(X, A) \otimes \check{\mathrm{H}}^*(Y, B) \to \check{\mathrm{H}}^*(X \times Y; A \otimes B).$$

In particular, if $A = B = R$ is a ring, one gets a homomorphism

$$\check{\mathrm{H}}^*(X, R) \otimes_R \check{\mathrm{H}}^*(Y, R) \to \check{\mathrm{H}}^*(X \times Y; R).$$

Theorem

This homomorphism is an isomorphism if, for example, X is a CW complex with finitely many cells in each dimension, and $\check{\mathrm{H}}^*(X, R)$ is a free R-module (or more generally, flat).

Internal ring structure on cohomology

Suppose X is a space and R is a commutative ring. Then we have natural homomorphisms

$$\check{\mathrm{H}}^*(X, R) \otimes_R \check{\mathrm{H}}^*(X, R) \to \check{\mathrm{H}}^*(X \times X; R) \to \check{\mathrm{H}}^*(X, R).$$

This endows $\check{\mathrm{H}}^*(X, R)$ with the structure of a *graded commutative ring*. For whatever historical reasons, the multiplication in cohomology is denoted by the symbol $\cup$, and is called "cup product".

Example

$$\begin{aligned}
\check{\mathrm{H}}^*(S^m \times S^n, \mathbb{Z}) &\cong \mathbb{Z}[u_m, u_n]/(u_m^2, u_n^2) \\
\check{\mathrm{H}}^*(\mathbb{C}P^n, \mathbb{Z}) &\cong \mathbb{Z}[u_2]/(u_2^{n+1}). \\
\check{\mathrm{H}}^*(\mathbb{R}P^n, \mathbb{Z}/2) &\cong \mathbb{Z}[x]/(x^{n+1}).
\end{aligned}$$

Applications: The Borsuk-Ulam theorem, non-existance of division algebra structure on $\mathbb{R}^n$ for $n \neq 2^k$.