Characteristic Classes
Universal bundles

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Universal bundles

Definition

Recall that a principal G-bundle $u\colon EG \to BG$ is called *universal*, if EG is contractible.

Theorem

Suppose X is a paracompact space. Let $u\colon EG \to BG$ be a universal principal G-bundle. There is a bijection

$$\begin{array}{ccc} [X, BG] & \to & Prin_G(X)/_{Iso} \\ f & \mapsto & f^*(u) \end{array}$$

Remark

CW complexes are paracompact. Results of this type often have two proofs. One that works for all paracompact spaces, and one that is specifically for CW complexes. In the lecture notes, the result is proved for paracompact spaces. In this lecture, we will go over the proof for CW complexes.

Homotopy invariance

The key to the proof of the main theorem is the following result.

Theorem (Homotopy invariance)

Let $p\colon E \to B$ be a principal G-bundle. Let X be a paracompact space, and let $f, g\colon X \to B$ be two maps. If f and g are homotopic, then the bundles $f^*(p)$ and $g^*(p)$ are isomorphic.

The theorem tells us that the function

$$\begin{array}{ccc} [X, B] & \to & \mathrm{Prin}_G(X)/_{\mathrm{Iso}} \\ f & \mapsto & f^*(p) \end{array}$$

is *well-defined*.

Notice that in this theorem we do not assume that p is a universal bundle.

The key to proving homotopy invariance of pullbacks is the following lemma.

Lemma

Suppose we have a principal G-bundle, $p\colon E \to X \times I$, where X is a paracompact space. Let $p_t\colon E_t \to X$ be the restriction of p to $p^{-1}(X \times \{t\})$. There is an isomorphism of principal G-bundles over $X \times I$

$$\begin{tikzcd} E_0 \times I \arrow[rr, "\cong"] \arrow[dr, "p_0 \times I"'] & & E \arrow[dl, "p"] \\ & X \times I & \end{tikzcd}$$

that restricts to the identity over $X \times \{0\}$.

Proof of the lemma, for X a CW complex

Since every morphism of principal G-bundles is an isomorphism, it is enough to prove that there exists *a* morphism of principal bundles

$$\begin{tikzcd}
E_0 \times I \arrow[r, dashed] \arrow[d] & E \arrow[d] \\
X \times I \arrow[r, "="] & X \times I
\end{tikzcd}$$

We will consider the more general problem of constructing a morphism of principal bundles. Suppose we have principal G-bundles $p_A \colon E_A \to A$, $p_B \colon E_B \to B$, and a map $f : A \to B$. We are interested in constructing the dashed arrow in the following diagram

$$\begin{tikzcd}
E_A \arrow[r, dashed] \arrow[d, "p_A"'] & E_B \arrow[d, "p_B"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

The next lemma gives a condition when the domain of the dashed map can be enlarged.

Lemma

Suppose that $A = A_1 \cup A_2$, A_1, A_2 are closed subsets of A, and we have a diagram of the following form:

$$
\begin{tikzcd}
 & E_A \arrow[dd, "p_A"] \arrow[dr, dashed] & \\
p^{-1}(A_1) \arrow[ur, hook] \arrow[rr] \arrow[dd, "p"'] & & E_B \arrow[dd, "p_B"] \\
 & A_1 \cup A_2 \arrow[dr, "f"] & \\
A_1 \arrow[ur, hook, "i"] \arrow[rr, "f \circ i"] & & B
\end{tikzcd}
$$

Suppose that p_A is trivializable over A_2, that p_B is trivializable over $f(A_2)$, and that A_2 retracts onto $A_1 \cap A_2$. Then the dashed arrow exists.

Proof

The problem is equivalent to the following one

$$
\begin{tikzcd}
 & G \times A_2 \arrow[dd] \arrow[dr, dashed] & & \\
G \times (A_1 \cap A_2) \arrow[ur, hook] \arrow[rr] \arrow[dd] & & G \times f(A_2) \arrow[r] \arrow[dd] & E_B \arrow[dd, "p"] \\
 & A_2 \arrow[dr, "f"] & & \\
A_1 \cap A_2 \arrow[ur, hook, "i"] \arrow[rr, "fi"] & & f(A_2) \arrow[r] & X \times I
\end{tikzcd}
$$

In this diagram, constructing the dashed arrow, is equivalent to constructing a map $A_2 \to G$, that extends a prescribed map $A_1 \cap A_2 \to G$. Since A_2 retracts onto $A_1 \cap A_2$, such a map exists.

Proof of the homotopy invariance lemma for $X = I^n$

Suppose that we have two principal G-bundles $p_i \colon E_i \to I^n \times I$, where $i = 1, 2$, and an isomorphism of bundles $p_1|_{I^n\times 0} \cong p_2|_{I^n\times 0}$. We want to prove that there is an (iso)morphism of bundles extending the isomorphism over $I^n \times \{0\}$.

$$\begin{tikzcd} E_1 \arrow[r, dashed] \arrow[d, "p_1"] & E_2 \arrow[d, "p_2"] \\ I^n \times I \arrow[r, "="] & I^n \times I \end{tikzcd}$$

By compactness, one can find a partition of I, $0 = t_0 < t_1 < t_2 < \cdots < t_k = 1$, such that p_1 and p_2 are both trivializable over each subcube of the form $[t_{i_1}, t_{i_1+1}] \times [t_{i_2}, t_{i_2+1}] \times \cdots \times [t_{i_{n+1}}, t_{i_{n+1}+1}]$. It is not hard to show that the subcubes can be ordered, $C_1, C_2, \ldots, C_{k^{n+1}}$, in such a way that each cube C_l (deformation) retracts on the intersection of C_l with the union of all the earlier cubes on the list and $I^n \times \{0\}$. By the previous lemma, the map of bundles can be constructed, cube by cube.

Some consequences

Corollary

Every principal G-bundle over I^n is trivial, i.e. is isomorphic to $G \times I^n$.

Corollary

Suppose we have principal bundles $p_1 \colon E_1 \to I^n \times I$, $p_2 \colon E_2 \to X$, a map $f \colon I^n \times I \to X$, and a lift of f over $I^n \times \{0\} \cup \partial I^n \times I$. The dashed map of principal G-bundles in the following diagram exists

$$\begin{tikzcd}
& E_1 \arrow[dd, "p_1"] \arrow[dr, dashed] & \\
p_1^{-1}(I^n \times \{0\} \cup \partial I^n \times I) \arrow[ur, hook] \arrow[rr] \arrow[dd, "p_1"'] & & E_2 \arrow[dd, "p_2"] \\
& I^n \times I \arrow[dr, "f"] & \\
I^n \times \{0\} \cup \partial I^n \times I \arrow[ur, hook] \arrow[rr, "f|_{I^n \times \{0\} \cup \partial I^n \times I}"] & & X
\end{tikzcd}$$

Homotopy invariance when X is a CW complex

Suppose X is a CW complex, and we have a principal G-bundle $p\colon E \to X \times I$. We want to prove that for every subcomplex $B \subset X$, there exists a morphism of principal G-bundles

$$\begin{tikzcd}
E_0 \cup p^{-1}(B \times \{0\}) \times I \arrow[r, dashed] \arrow[d] & E \arrow[d, "p"] \\
X \times \{0\} \cup B \times I \arrow[r, hook, "i"] & X \times I
\end{tikzcd}$$

The key step is to prove that if the dashed arrow can be constructed for some subcomplex $B \subset X$, then it can be extended for a subcomplex of the form $B \cup_\phi D^n \subset X$, where D^n is some cell of X that is attached to B via a map $\phi\colon S^{n-1} \to B$. Let $\Psi\colon D^n \to X$ be the characteristic map of D^n.

We need to find the dashed arrow in the following diagram.

$$\begin{tikzcd}
& E_0 \cup p^{-1}(B \cup_\phi D^n \times \{0\}) \times I \arrow[dd] \arrow[dr, dashed] & \\
E_0 \cup p^{-1}(B) \times I \arrow[ur, hook] \arrow[rr] \arrow[dd] & & E \arrow[dd, "p"] \\
& X \times \{0\} \cup (B \cup_\phi D^n) \times I \arrow[dr, hook] & \\
X \times \{0\} \cup B \times I \arrow[ur, hook, "i"] \arrow[rr, hook] & & X \times I
\end{tikzcd}$$

This is equvalent to finding the dashed arrow here:

$$\begin{tikzcd}
& \Psi^*(p_0) \times I \arrow[dd] \arrow[dr, dashed] & \\
\Psi^*(p_0) \cup \phi^*(p_0) \times I \arrow[ur, hook] \arrow[rr] \arrow[dd] & & E \arrow[dd, "p"] \\
& D^n \times I \arrow[dr, "\Psi \times I"] & \\
D^n \times \{0\} \cup S^{n-1} \times I \arrow[ur, hook, "i"] \arrow[rr, hook, "(\Psi \times I) \circ i"] & & X \times I
\end{tikzcd}$$

This is possible by a result we just proved.

Back to universal bundles

Now suppose again that $u\colon EG \to BG$ is a universal principal G-bundle. We know now that pullback induces a well-defined function

$$[X, BG] \to \mathrm{Prin}_G(X)/_{\mathrm{Iso}}.$$

It remains to prove that this function is a bijection. To do this, we will construct an inverse to this function. Thus, given a principal G-bundle $p\colon E \to X$, we want to construct a map $f\colon X \to BG$, such that $p \cong f^*(u)$.

Lemma

Constructing a map f as above is equivalent to constructing a G-equivariant map $\tilde{f}\colon E \to EG$.

Lemma

Suppose that X is a CW complex, and $p\colon E \to X$ is a principal G-bundle. There exists a G-equivariant map $E \to EG$, and any two such maps are G-equivariantly homotopic.

Proof.

The proof uses induction on cells of X, and the contractibility of EG. Suppose that for some subcomplex $B \subset X$, we have constructed a G-equivariant map $p^{-1}(B) \to EG$. Suppose that D^n is a cell of X that is attached to B via a map $\phi\colon S^{n-1} \to B$. We need to fill in the dashed arrow in the following diagram of G-maps

$$\begin{tikzcd} G \times S^{n-1} \arrow[r] \arrow[d] & G \times D^n \arrow[d, dashed] \\ p^{-1}(B) \arrow[r] & EG \end{tikzcd}$$

A G-equivariant map $G \times D^n \to EG$, extending a given map $G \times S^{n-1} \to EG$ is the same thing as a non-equivariant map $D^n \to EG$, extending a given map $S^{n-1} \to EG$. Such a map exists because EG is contractible.

Uniqueness

Need to show that any two G-equivariant maps $f_0, f_1 \colon E \to EG$ are G-homotopic. The idea is to construct, for each subcomplex $B \subset X$, a G-equivariant map

$$H_B \colon E \times \{0,1\} \cup p^{-1}(B) \times I \to EG$$

extending the map $E \times \{0,1\} \to EG$ induced by f_0, f_1.
Suppose by induction that we have constructed such a map for some $B \subset X$, and we want to extend it over $B \cup D^n$, where D^n is some cell of X attached to B. This means that we need to extend H_B to a map of the following form

$$H_B \colon E \times \{0,1\} \cup p^{-1}(B \cup D^n) \times I \to EG$$

To solve the problem one needs to construct a G-map

$$p^{-1}(D^n \times I) \cong G \times D^n \times I \to EG$$

That is predetermined on

$$P^{-1}(D^n \times \partial I \cup \partial D^n \times I) \cong G \times \partial(D^n \times I).$$

Once again, such a map exists by contractibility of EG.