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3. A subset $F \subseteq X$ is closed if and only if $F \cap \overline{e}_\alpha$ if closed (in X) for each cell e_α

CW-complexes

form a large class of topological spaces of combinatorial nature which makes them very nice to work with in homotopy theory

Examples

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1. $\bullet = e^{\circ}$

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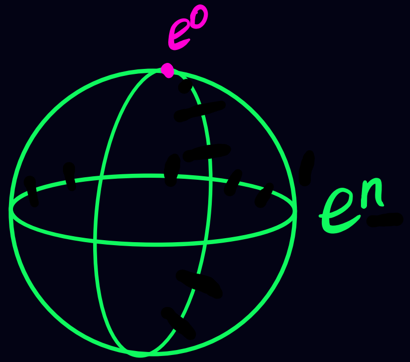
2. $S^n = e^0 \sqcup e^n$



Examples

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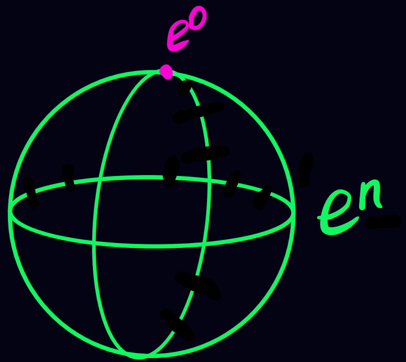
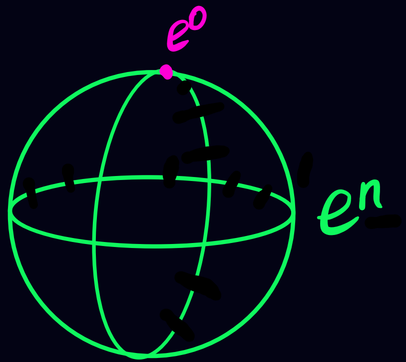


2. $S^n = e^0 \sqcup e^n$

3. $S^0 = \bullet \quad \bullet = e_1^0 \sqcup e_2^0$

Examples



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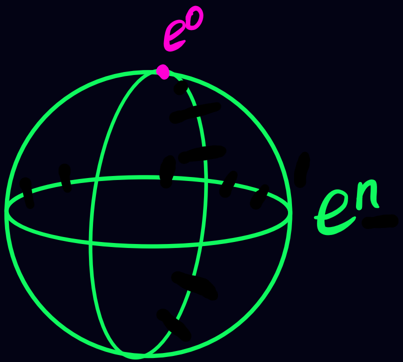

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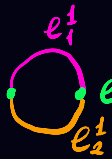
3. $S^0 = \bullet \quad \bullet = e^0_1 \sqcup e^0_2$, $S^1 = e^0_1$ 

$e^0_2 = e^0_1 \sqcup e^0_2 \sqcup e^1_1 \sqcup e^1_2, \ldots, S^n = \bigsqcup_{i=0}^{n} e^i_1 \sqcup e^i_2$

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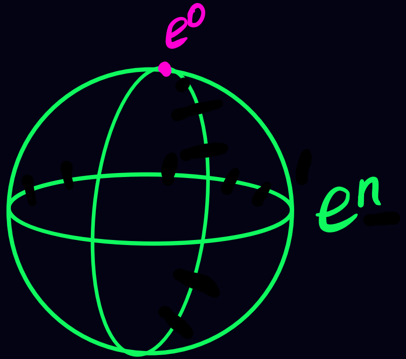

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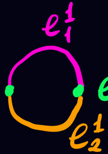
3. $S^0 = \bullet \quad \bullet = e^0_1 \sqcup e^0_2$, $S^1 = e^0_1$  $e^0_2 = e^0_1 \sqcup e^0_2 \sqcup e^1_1 \sqcup e^1_2, \dots, S^n = \bigsqcup_{i=0}^{n} e^i_1 \sqcup e^i_2$

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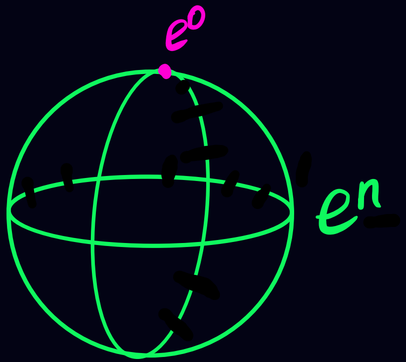

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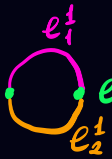
3. $S^0 = \bullet \quad \bullet = e_1^0 \sqcup e_2^0$, $S^1 = e_1^0$ 

$e_2^0 = e_1^0 \sqcup e_2^0 \sqcup e_1^1 \sqcup e_2^1, \ldots, S^n = \bigsqcup_{i=0}^{n} e_1^i \sqcup e_2^i$

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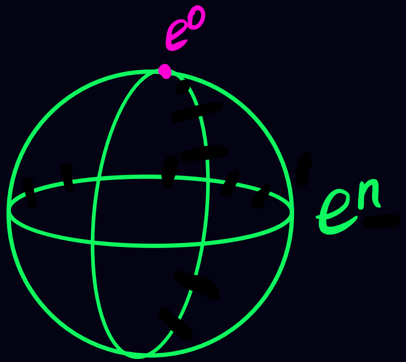

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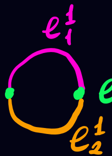
3. $S^0 = \bullet \quad \bullet = e_1^0 \sqcup e_2^0$, $S^1 = e_1^0$ 

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Another definition

Every CW-complex X has the skeletal filtration

$$X^0 \subseteq X^1 \subseteq \dots \subseteq X^m \subseteq \dots \subseteq X = \bigcup_{i \geq 0} X^i$$

such the following diagram is a pushout diagram in Top

$$\begin{tikzcd}
\coprod_\alpha \partial D^n \arrow[r, phantom, "\subseteq"] \arrow[d, "(\phi^n_\alpha|_{\partial D^n})_\alpha"'] & \coprod_\alpha D^n \arrow[d, "(\phi^n_\alpha)_\alpha"] \\
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cells of dim $\leq m$

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Another definition

The convers is also true

Given a topological space X with filtration

$$X^0 \subseteq X^1 \subseteq \dots \subseteq X^m \subseteq \dots$$

such that:

1. X^0 has discrete topology
2. X^n is an adjunction of n-disks to X^{n-1}, i.e. $X^n \cong X^{n-1} \coprod_\alpha D^n / \sim$
3. $X = \bigcup_{i \geq 0} X^i$ has the colimit topology, i.e.

$$F \subseteq X \text{ is closed} \iff F \cap X^n \text{ is closed in } X^n \text{ for all } n \geq 0$$

Then X is Hausdorff and posses a unique CW-structure such that X^n is the n-skeleton of X

Grassmanians and Schubert cells

We know that $Gr_1(\mathbb{R}^N) = \mathbb{R}P^{N-1}$ $\longleftarrow$ CW-complex !

Is there a CW-structure on $Gr_n(\mathbb{R}^N)$? What does it look like?

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Schubert varieties = $\overline{\text{Schubert cells}}$

Grassmanians and Schubert cells

Fix a complete flag of subspaces in $\mathbb{R}^N$: $\quad 0 = F_0 \subset F_1 \subset \dots \subset F_N = \mathbb{R}^N$

take any $W \in Gr_n(\mathbb{R}^N)$ and consider another flag $\quad 0 = W \cap F_0 \subseteq W \cap F_1 \subseteq \dots \subseteq W \cap F_N = W$

Let $1 \leq a_1 < a_2 < \dots < a_n \leq N$ be the indices such that $\dim(W \cap F_{a_i}) = \dim(W \cap F_{a_i - 1}) + 1$

Define $\quad e(a_1, a_2, \dots, a_n) := \{ W \in Gr_n(\mathbb{R}^N) \mid \dim(W \cap F_{a_i}) = \dim(W \cap F_{a_i - 1}) + 1 \}$

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