The tangent bundle of a smooth manifold or tangent sheaf of an algebraic variety (Chapter 2.5.2.)

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Derivation

Let us recall the algebraic notion of derivation.

Definition

Let A be a ring, R an A-algebra and M an R-module. Then an A-derivation $D : R \to M$ is an A-linear morphism, which satisfies the Leibniz rule

$$D(ab) = aD(b) + bD(a), \forall a, b \in R.$$

We denote the set of these by $Der_A(R, M)$. It has a natural structure of A-module.

Example

Let $M = R = C^\infty(\mathbb{R}^n)$ be the algebra of smooth real-valued functions on $\mathbb{R}^n$. For a chosen basis of $\mathbb{R}^n$, let $x_i : \mathbb{R}^n \to \mathbb{R}$ be the coordinate function $(x_1, ..., x_n) \to x_i$. Then $\frac{\partial}{\partial x_i}$ is a derivation on R.

Tangent space

Definition

let X be a variety (or a smooth manifold) and $x \in X$. Denote by $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_{X,x}$. We call $\mathfrak{m}/\mathfrak{m}^2$ the cotangent space of X at x, and define its algebraic dual $T_xX := \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$ to be the tangent space of X at x. Also, we say x is a smooth point of X if $\dim T_xX = \dim X$.

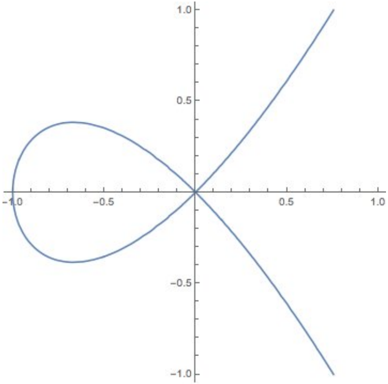
Example

Let $X = \mathbb{A}^2$ be the affine plane and let $a = (0,0)$. Then $\mathcal{O}_{X,a} = k[x,y]_{(x,y)} = \{\frac{f}{g} | f, g \in k[x,y] \textit{ and } g((0,0)) \neq 0\}$ which is the localization of $k[x,y]$ at the ideal (x,y). The unique maximal ideal corresponds to the ideal (x,y). Thus, $\mathfrak{m}/\mathfrak{m}^2 \cong \frac{(x,y)}{(x,y)^2} = k^{\oplus 2}$ and

$$\mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k) = \mathrm{Hom}_k(k^{\oplus 2}, k) \cong k^{\oplus 2}$$

Tangent space of nodal curve

Let $X = V(y^2 - x^3 - x^2)$ be a nodal elliptic curve embedded in $\mathbb{A}^2$. We want to calculate the tangent space of X at origin.



Tangent space of nodal curve

- First, we see the coordinate ring is $A(X) = \frac{k[x,y]}{(y^2-x^3-x^2)}$, which corresponds to functions that can define on the space X.
- Now, we can make the stalk of structure sheaf $\mathcal{O}_{X,a}$ for $a = (0,0)$, which is the localization of the ring $A(X)$ at origin a.
- The unique maximal ideal $\mathfrak{m}$ is the localization of $\overline{(x,y)}$.
- In the next step, $\mathfrak{m}/\mathfrak{m}^2$ is localization of $\frac{(x,y)}{y^2-x^3-x^2} / \frac{(x^2,y^2,xy)}{y^2-x^3-x^2}$, which is isomorphic to $\frac{(x,y)}{x^2,xy,y^2} \cong k^{\oplus 2}$. This shows $\dim_k T_{(0,0)}X = 2$ despite the fact that X is one dimensional.

Generators of maximal ideal of $C_0^\infty(\mathbb{R}^n)$

Lemma

Let $f \in C^\infty(\mathbb{R}^n)$ then for every $a \in \mathbb{R}^n$, $f(x) = f(a) + \sum_{i=1}^n g_i(x)(x_i - a_i)$ for some smooth function $g_i(x)$.

We can write $f(x) - f(a)$ as

$$\int_0^1 \frac{d}{dt} f(a + t(x-a))dt = \sum \int_0^1 \frac{\partial f}{\partial x_i}(a + t(x-a))(x_i - a_i)dt$$

$$= \sum (x_i - a_i) \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(a + t(x-a))}_{=g_i(x)}$$

Proposition

The maximal ideal $C_0^\infty(\mathbb{R}^n)$ is finitely generated by $\{x_1, ..., x_n\}$

Some basic algebra facts

For a commutative ring R, we define $\dim R$ is the maximum length of chain of prime ideals of R.

$$\dim(R) = \max(\{n \mid \mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_n \textit{ such that } \mathfrak{p}_i \textit{ are prime ideal}\})$$

Also, We define the height of a prime ideal $\mathfrak{p}$ to be the maximum length of descending chain of prime ideals from $\mathfrak{p}$.

Some important theorems:

- Let X be a variety (or a smooth manifold) then $\dim X = \dim \mathcal{O}_{X,x}$ for every $x \in X$.
- Krull dimension theorem: Let R be Noetherian ring, and let prime ideal $\mathfrak{p}$ is generated by d elements then $\mathrm{ht}(\mathfrak{p}) \leq d$.
- Nakayama lemma: Let R be a local Noetherian ring with the maximal ideal $\mathfrak{m}$ then the minimum number of generator of $\mathfrak{m}$ is equal to $\dim_k \mathfrak{m}/\mathfrak{m}^2$, where $k = R/\mathfrak{m}$

Dimension of tangent space

Proposition

Let X be a variety (or a smooth manifold) then we have the followings:

1. $\dim_k T_x X \geq \dim X$, *and for a smooth manifold, always equality happens.*
2. *We endow k with an $\mathcal{O}_{X,x}$-module structure via the following law :*

$$f.\alpha = f(x).\alpha$$

for $f \in \mathcal{O}_{X,x}$ and $\alpha \in k$. Then $T_x X \cong \mathrm{Der}_k(\mathcal{O}_{X,x}, k)$.

Sketch of proof

Proof.

Part a)

- We want to prove $d := \dim_{\mathrm{k}} \mathrm{Hom}(\mathfrak{m}/\mathfrak{m}^2, k) = \dim_k \mathfrak{m}/\mathfrak{m}^2 \geq \dim \mathcal{O}_{X,x} = \dim X$
- Since $\mathfrak{m}$ is **finitely generated**, by Nakayama's lemma we have $\mathfrak{m}$ is generated by d elements.By Krull theorem,

$$\dim \mathcal{O}_{X,x} = \mathrm{ht}\, \mathfrak{m} \leq d.$$

Note that for a smooth manifold of dimension n, we have

$$\mathfrak{m}/\mathfrak{m}^2 = (x_1, ..., x_n)/(x_1^2, x_1x_2, .., x_ix_j, ..., x_n^2) \cong \mathbb{R}^n$$

□

Proof

Proof.

Part b)

- Let $D \in \mathrm{Der}_k(\mathcal{O}_{X,x}, k)$. If $f, g \in \mathfrak{m}$, then $D(fg) = g(x)D(f) + f(x)D(g) = 0$, thus $D|_{\mathfrak{m}} \in T_xX$
- Let $\phi : \mathfrak{m}/\mathfrak{m}^2 \to k$, then we want to construct a map $D_\phi : \mathcal{O}_{X,x} \to k$. We claim that the appropriate choice is $D_\phi(f) = \phi(f_0)$,where $f = f_0 + k$ and $f_0 \in \mathfrak{m}$.

One can check that $D \to D|_{\mathfrak{m}} \to D_{D|_{\mathfrak{m}}}$ is identity and so is $\phi \to D_\phi \to D_\phi|_{\mathfrak{m}}$. $\square$

Corollary

Let $TM := \bigcup_{a\in M} T_aM$, and $\pi = TM \to M$ be the natural projection. For a smooth chart (U, ϕ), we have

$$\Phi : \pi^{-1}(U) \to \mathbb{R}^{2n} : \ \sum v^i \frac{\partial}{\partial \phi_i}|_p \to (\phi(p), v)$$

which is a bijection onto its image, so TM is a real vector bundle.

Corollary

For an n-dimensional manifold M, TM is an n-dimensional real vector bundle.

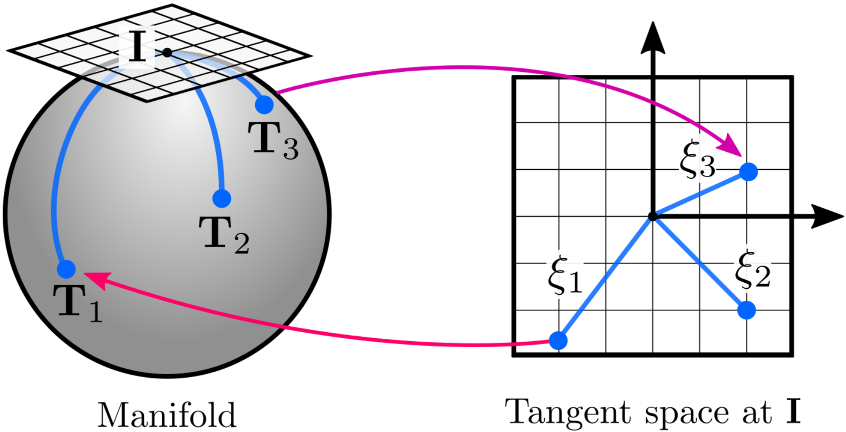
Cotangent sheaf

Let X be a variety (or a smooth manifold) over k, and let $\Delta : X \to X \times_k X$ be the diagonal map. we define the cotangent sheaf (Kähler differential) $\Omega_X = \Delta^*(\mathcal{I}/\mathcal{I}^2)$. One can see that the fiber at each point $x \in X$, is

$$\Omega_{\mathcal{O}} \otimes k \cong \mathfrak{m}/\mathfrak{m}^2;$$

see Hartshorne, chapter 2, section 8.

Tangent space by curve



Tangent space by curve

Definition

Let $M \subset \mathbb{R}^n$ be a smooth submanifold. Define a vector bundle $T^g M \to M$ (the "g" stands for "geometric") by

$$T^g M = \left\{ (m, v) \in M \times \mathbb{R}^N \ s.t. \ \begin{array}{c} v = \gamma'(0) \ \textit{for some smooth curve} \\ \gamma : (-1, 1) \to M \ \textit{with} \ \gamma(0) = m \end{array} \right\}.$$

Let $p : \mathbb{R}^N \to V$ be a projection to n-dimensional subspace of $\mathbb{R}^N$ s.t.

- $p(x) = 0$
- p is a chart in a neighborhood U of x in M. Now, we can define isomorphism

$$\tilde{p} : T^g M|_U \to V \times V; \ \tilde{p}(m, v) = (p(m), p(v)).$$

We now define

$$\Phi : \Gamma_{T^g M}(U) \to TM(U)$$

by $\Phi(s)(f)(x) = D_{s(x)} f$ for $s : U \to T^g M|_U, f \in C_U^\infty(V)$ for $V \subset U$ open, and $x \in V$.

Proposition

The map Φ is an isomorphism of vector bundles.

Proof.

- We can assume $M = \mathbb{R}^n$
- It is enough to prove it on the stalks, so we aim to prove

$$\Phi_x : \{\gamma'(0) | \gamma : (-1, 1) \to \mathbb{R}^n, \gamma(0) = 0\} = \mathbb{R}^n \to \mathrm{Der}_{\mathbb{R}}((C^\infty_{\mathbb{R}^n})^+, \mathbb{R})$$

is an isomorphism.

$\square$

Thank you!