Operations on vector bundles

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Overview

Goal:
Generalize notions such as $\oplus, \otimes, V^\vee, \Lambda, \mathrm{Hom}$ to vector bundles

Summary

- Define tensor product as a functor.
- Introduce continuous functors on $Vect_k$.
- Show they induce to functors on $Vect(B)$.

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Tensor product of two factors,

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It's functorial since

$$(A \otimes B) \circ (A' \otimes B') = (AA' \otimes BB')$$

Tensor Product

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Repeating this for the first factor shows that $(A, B) \mapsto A \otimes B$ is a bilinear map, therefore continuous!

Definition. An (p, q)-ary continuous functor of k-vector spaces is a functor

$$F : (Vect_k)^p \times (Vect_k^{\mathrm{op}})^q \to Vect_k$$

from (p + q)-tuples of finite-dimensional k-vector spaces to finite-dimensional k-vector spaces which is continuous in the following sense:

Continuous functor

For

$$V, W \in (Vect_k)^p \times (Vect_k^{\mathrm{op}})^q$$

$$\mathrm{Hom_k}(V, W) \xrightarrow{F} \mathrm{Hom_k}(F(V), F(W))$$

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We showed this is true for

$$(A, B) \to A \otimes B$$

Proposition 2.4.10. Given any (p,q)-ary continuous functor F of k-vector spaces, there is an induced functor

$$F : Vect_k(B)^p \times (Vect_k(B)^{\mathrm{op}})^q \to Vect_k(B)$$

which acts like F on each fiber of B

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Generally, given $E_1, \ldots, E_{p+q}$, define

$$F(E_1, \ldots, E_{p+q}) = \bigsqcup_{b \in B} F((E_1)_b, \ldots, (E_{p+q})_b)$$

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Now we need a topology which allows for local trivializations!

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We have local trivializations for E and E',

$$\tilde{\phi}^{\alpha} : k^n \times U_\alpha \to E_{U_\alpha}, \quad \tilde{\psi}^{\alpha} : k^m \times U_\alpha \to E'_{U_\alpha}$$

whose restrictions to the fibers are denoted by $\phi^{\alpha}_b, \psi^{\alpha}_b$.

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$$\begin{aligned} \Phi^\alpha : (k^n \otimes k^m) \times U_\alpha &\to (E \otimes E')_{U_\alpha} \\ (v \otimes w, b) &\mapsto (\phi^\alpha_b \otimes \psi^\alpha_b)(v \otimes w) \end{aligned}$$

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$$\begin{aligned} \Phi^\alpha : F(k^{n_1}, \dots, k^{n_{p+q}}) \times U_\alpha &\to F(E_1, \dots, E_{p+q})|_{U_\alpha} \\ (y, b) &\mapsto F(\phi_{b,1}, \dots, \phi_{b,p+q}^{-1})(y) \end{aligned}$$

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This is the step where we use the continuity of the functor! Our cocycles are exactly

$$(\phi_b^{-\alpha}\phi_b^{\beta} \otimes \psi_b^{-\alpha}\psi_b^{\beta}) = c_{\alpha\beta} \otimes c'_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(k^n \otimes k^m)$$

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In general, the cocycles are given by

$$F\left(c_{1,\alpha\beta}, \ldots, c_{p+q,\alpha\beta}^{-1}\right) : U_{\alpha\beta} \to \mathrm{GL}(F(k^{n_1}, \ldots, k^{n_{p+q}}))$$

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Lastly, for any morphism of vector bundles f, g, with f_b, g_b being the restriction to the fibers we can consider

$$f \otimes g = \bigsqcup_{b \in B} f_b \otimes g_b$$

Clearly, respects the vectorspace structure but why is it continuous?

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In general

$$F(f_1, \dots, f_{p+q}) = \bigsqcup_{b \in B} F(f_{b,1}, \dots, f_{b,p+q})$$

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Picard group

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This allows us to define the $\mathrm{Pic}_k(B)$ as the set of isomorphism classes of vector bundles with tensorproduct.

End

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