It should be noted that the integer $b - a$ has been defined only when $b > a$. The interpretation of the symbol $b - a$ as a *negative integer* when $b < a$ will be discussed later (p. 54 et seq.).

It is often convenient to use one of the notations, $b \geq a$ (read, "b is greater than or equal to a") or $a \leq b$ (read, "a is less than or equal to b"), to express the denial of the statement, $a > b$. Thus, $2 \geq 2$, and $3 \geq 2$.

We may slightly extend the domain of positive integers, represented by boxes of dots, by introducing the integer *zero*, represented by a completely empty box. If we denote the empty box by the usual symbol 0, then, according to our definition of addition and multiplication,

$$a + 0 = a,$$
$$a \cdot 0 = 0,$$

for every integer a. For $a + 0$ denotes the addition of an empty box to the box a, while $a \cdot 0$ denotes a box with no columns; i.e. an empty box. It is then natural to extend the definition of subtraction by setting

$$a - a = 0$$

for every integer a. These are the characteristic arithmetical properties of zero.

Geometrical models like these boxes of dots, such as the ancient abacus, were widely used for numerical calculations until late in the middle ages, when they were slowly displaced by greatly superior symbolic methods based on the decimal system.

2. The Representation of Integers

We must carefully distinguish between an integer and the symbol, 5, V, $\cdots$, etc., used to represent it. In the decimal system the ten digit symbols, 0, 1, 2, 3, $\cdots$, 9, are used for zero and the first nine positive integers. A larger integer, such as "three hundred and seventy-two," can be expressed in the form

$$300 + 70 + 2 = 3 \cdot 10^2 + 7 \cdot 10 + 2,$$

and is denoted in the decimal system by the symbol 372. Here the important point is that the meaning of the digit symbols 3, 7, 2 depends on their *position* in the units, tens, or hundreds place. With this "positional notation" we can represent any integer by using only the ten digit symbols in various combinations. The general rule is to express an integer in the form illustrated by

$$z = a \cdot 10^3 + b \cdot 10^2 + c \cdot 10 + d,$$

where the digits a, b, c, d are integers from zero to nine. The integer z is then represented by the abbreviated symbol

$$abcd.$$

We note in passing that the coefficients d, c, b, a are the remainders left after successive divisions of z by 10. Thus

$$\begin{array}{rl} 10\overline{)372} & \text{Remainder} \\ 10\overline{)37} & 2 \\ 10\overline{)3} & 7 \\ 0 & 3 \end{array}$$

The particular expression given above for z can only represent integers less than ten thousand, since larger integers will require five or more digit symbols. If z is an integer between ten thousand and one hundred thousand, we can express it in the form

$$z = a \cdot 10^4 + b \cdot 10^3 + c \cdot 10^2 + d \cdot 10 + e,$$

and represent it by the symbol *abcde*. A similar statement holds for integers between one hundred thousand and one million, etc. It is very useful to have a way of indicating the result in perfect generality by a single formula. We may do this if we denote the different coefficients, $e, d, c, \cdots$, by the single letter a with different "subscripts," $a_0, a_1, a_2, a_3, \cdots$, and indicate the fact that the powers of ten may be as large as necessary by denoting the highest power, not by 10^3 or 10^4 as in the examples above, but by 10^n, where n is understood to stand for an arbitrary integer. Then the general method for representing an integer z in the decimal system is to express z in the form

$$(1) \qquad z = a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \cdots + a_1 \cdot 10 + a_0,$$

and to represent it by the symbol

$$a_n a_{n-1} a_{n-2} \cdots a_1 a_0.$$

As in the special case above, we see that the digits $a_0, a_1, a_2, \cdots, a_n$ are simply the successive remainders when z is divided repeatedly by 10.

In the decimal system the number ten is singled out to serve as a base. The layman may not realize that the selection of ten is not essential, and that any integer greater than one would serve the same purpose. For example, a *septimal* system (base 7) could be used. In such a system, an integer would be expressed as

$$(2) \qquad b_n \cdot 7^n + b_{n-1} \cdot 7^{n-1} + \cdots + b_1 \cdot 7 + b_0,$$

where the b's are digits from zero to six, and denoted by the symbol

$$b_n b_{n-1} \cdots b_1 b_0 .$$

Thus "one hundred and nine" would be denoted in the septimal system by the symbol 214, meaning

$$2 \cdot 7^2 + 1 \cdot 7 + 4.$$

As an exercise the reader may prove that the general rule for passing from the base ten to any other base B is to perform successive divisions of the number z by B; the remainders will be the digits of the number in the system with base B. For example:

	Remainder
7)109	
7)15	4
7)2	1
0	2

109 (decimal system) = 214 (septimal system).

It is natural to ask whether any particular choice of base would be most desirable. We shall see that too small a base has disadvantages, while a large base requires the learning of many digit symbols, and an extended multiplication table. The choice of twelve as base has been advocated, since twelve is exactly divisible by two, three, four, and six, and, as a result, work involving division and fractions would often be simplified. To write any integer in terms of the base twelve (duodecimal system), we require two new digit symbols for ten and eleven. Let us write α for ten and β for eleven. Then in the duodecimal system "twelve" would be written 10, "twenty-two" would be 1α, "twenty-three" would be 1β, and "one hundred thirty-one" would be $\alpha\beta$.

The invention of positional notation, attributed to the Sumerians or Babylonians and developed by the Hindus, was of enormous significance for civilization. Early systems of numeration were based on a purely additive principle. In the Roman symbolism, for example, one wrote

$$\text{CXVIII} = \text{one hundred} + \text{ten} + \text{five} + \text{one} + \text{one} + \text{one}.$$

The Egyptian, Hebrew, and Greek systems of numeration were on the same level. One disadvantage of any purely additive notation is that more and more new symbols are needed as numbers get larger. (Of course, early scientists were not troubled by our modern astronomical or atomic magnitudes.) But the chief fault of ancient systems, such as the Roman, was that computation with numbers was so difficult that only the specialist could handle any but the simplest problems. It is quite different with the Hindu positional system now in use. This was introduced into medieval Europe by the merchants of Italy, who learned it from the Moslems. The positional system has the agreeable property that all numbers, however large or small, can be represented by the use of a small set of different digit symbols (in the decimal system, these are the "Arabic numerals" 0, 1, 2, $\cdots$, 9). Along with this goes the more important advantage of ease of computation. The rules of reckoning with numbers represented in positional notation can be stated in the form of addition and multiplication tables for the digits that can be memorized once and for all. The ancient art of computation, once confined to a few adepts, is now taught in elementary school. There are not many instances where scientific progress has so deeply affected and facilitated everyday life.

3. Computation in Systems Other than the Decimal

The use of ten as a base goes back to the dawn of civilization, and is undoubtedly due to the fact that we have ten fingers on which to count. But the number words of many languages show remnants of the use of other bases, notably twelve and twenty. In English and German the words for 11 and 12 are not constructed on the decimal principle of combining 10 with the digits, as are the "teens," but are linguistically independent of the words for 10. In French the words "vingt" and "quatre-vingt" for 20 and 80 suggest that for some purposes a system with base 20 might have been used. In Danish the word for 70, "halvfirsindstyve," means half-way (from three times) to four times twenty. The Babylonian astronomers had a system of notation that was partly sexagesimal (base 60), and this is believed to account for the customary division of the hour and the angular degree into 60 minutes.

In a system other than the decimal the rules of arithmetic are the same, but one must use different tables for the addition and multiplication of digits. Accustomed to the decimal system and tied to it by the number words of our language, we might at first find this a little confusing. Let us try an example of multiplication in the septimal system. Before proceeding, it is advisable to write down the tables we shall have to use:

Addition

	1	2	3	4	5	6
1	2	3	4	5	6	10
2	3	4	5	6	10	11
3	4	5	6	10	11	12
4	5	6	10	11	12	13
5	6	10	11	12	13	14
6	10	11	12	13	14	15

Multiplication

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	11	13	15
3	3	6	12	15	21	24
4	4	11	15	22	26	33
5	5	13	21	26	34	42
6	6	15	24	33	42	51

Let us now multiply 265 by 24, where these number symbols are written in the septimal system. (In the decimal system this would be equivalent to multiplying 145 by 18.) The rules of multiplication are the same as in the decimal system. We begin by multiplying 5 by 4, which is 26, as the multiplication table shows.

$$\begin{array}{r} 265 \\ \underline{24} \\ 1456 \\ \underline{563} \\ 10416 \end{array}$$

We write down 6 in the units place, "carrying" the 2 to the next place. Then we find $4 \cdot 6 = 33$, and $33 + 2 = 35$. We write down 5, and proceed in this way until everything has been multiplied out. Adding $1{,}456 + 5{,}630$, we get $6 + 0 = 6$ in the units place, $5 + 3 = 11$ in the sevens place. Again we write down 1 and keep 1 for the forty-nines place, where we have $1 + 6 + 4 = 14$. The final result is $265 \cdot 24 = 10{,}416$.

To check this result we may multiply the same numbers in the decimal system. 10,416 (septimal system) may be written in the decimal system by finding the powers of 7 up to the fourth: $7^2 = 49$, $7^3 = 343$, $7^4 = 2{,}401$. Hence $10{,}416 = 2{,}401 + 4 \cdot 49 + 7 + 6$, this evaluation being in the decimal system. Adding these numbers we find that 10,416 in the septimal system is equal to 2,610 in the decimal system. Now we multiply 145 by 18 in the decimal system; the result is 2,610, so the calculations check.

Exercises: 1) Set up the addition and multiplication tables in the duodecimal system and work some examples of the same sort.

2) Express "thirty" and "one hundred and thirty-three" in the systems with the bases 5, 7, 11, 12.

3) What do the symbols 11111 and 21212 mean in these systems?

4) Form the addition and multiplication tables for the bases 5, 11, 13.

From a theoretical point of view, the positional system with the base 2 is singled out as the one with the smallest possible base. The only digits in this *dyadic system* are 0 and 1; every other number z is represented by a row of these symbols. The addition and multiplication tables consist merely of the rules $1 + 1 = 10$ and $1 \cdot 1 = 1$. But the disadvantage of this system is obvious: long expressions are needed to represent small numbers. Thus seventy-nine, which may be expressed as $1 \cdot 2^6 + 0 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2 + 1$, is written in the dyadic system as 1,001,111.

As an illustration of the simplicity of multiplication in the dyadic system, we shall multiply seven and five, which are respectively 111 and 101. Remembering that $1 + 1 = 10$ in this system, we have

$$\begin{array}{r} 111 \\ \underline{101} \\ 111 \\ \underline{111} \\ 100011 \end{array} = 2^5 + 2 + 1,$$

which is thirty-five, as it should be.

Gottfried Wilhelm Leibniz (1646–1716), one of the greatest intellects of his time, was fond of the dyadic system. To quote Laplace: "Leibniz saw in his binary arithmetic the image of creation. He imagined that Unity represented God, and zero the void; that the Supreme Being drew all beings from the void, just as unity and zero express all numbers in his system of numeration."

Exercise: Consider the question of representing integers with the base a. In order to name the integers in this system we need words for the digits $0, 1, \cdots, a - 1$ and for the various powers of a: $a, a^2, a^3, \cdots$. How many different number words are needed to name all numbers from zero to one thousand, for $a = 2, 3, 4, 5, \cdots, 15$? Which base requires the fewest? (Examples: If $a = 10$, we need ten words for the digits, plus words for 10, 100, and 1000, making a total of 13. For $a = 20$, we need twenty words for the digits, plus words for 20 and 400, making a total of 22. If $a = 100$, we need 100 plus 1.)

*§2. THE INFINITUDE OF THE NUMBER SYSTEM. MATHEMATICAL INDUCTION

1. The Principle of Mathematical Induction

There is no end to the sequence of integers $1, 2, 3, 4, \cdots$; for after any integer n has been reached we may write the next integer, $n + 1$. We express this property of the sequence of integers by saying that there are *infinitely many* integers. The sequence of integers represents the simplest and most natural example of the mathematical infinite, which plays a dominant rôle in modern mathematics. Everywhere in this book we shall have to deal with collections or "sets" containing infinitely many mathematical objects, like the set of all points on a line or the set of all triangles in a plane. The infinite sequence of integers is the simplest example of an infinite set.

The step by step procedure of passing from n to $n + 1$ which generates the infinite sequence of integers also forms the basis of one of the most fundamental patterns of mathematical reasoning, the principle of

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