number theory. The main goal of finite group theory is to give a complete classification of all the "simple groups."

A major breakthrough occurred in 1963 when Walter Feit and John Thompson proved that every simple group is either cyclic or has an even number of elements. This had been conjectured by Burnside many years earlier. Following the inspiration of the Feit-Thompson success, a tremendous surge of new activity erupted in finite group theory. Today specialists in this area believe they are within a stone's throw of a complete classification of the simple groups.

Further Readings. See Bibliography

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The Prime Number Theorem

THE THEORY of numbers is simultaneously one of the most elementary branches of mathematics in that it deals, essentially, with the arithmetic properties of the integers 1, 2, 3, . . . and one of the most difficult branches insofar as it is laden with difficult problems and difficult techniques.

Among the advanced topics in theory of numbers, three may be selected as particularly noteworthy: the theory of partitions, Fermat's "Last Theorem," and the prime number theorem. The theory of partitions concerns itself with the number of ways in which a number may be broken up into smaller numbers. Thus, including the "null" partition, two may be broken up as 2 or 1 + 1. Three may be broken up as 3, 2 + 1, 1 + 1 + 1, four may be broken up as 4, 3 + 1, 2 + 2, 2 + 1 + 1, 1 + 1 + 1 + 1. The number of ways that a given number may be broken up is far from a simple matter, and has been the object of study since the mid-seventeen hundreds. The reader might like to experi-

Pierre de Fermat
1601–1665

ment and see whether he can systematize the process and verify that the number 10 can be broken up in 42 different ways.

Fermat's "Last Theorem" asserts that if $n > 2$, the equation $x^n + y^n = z^n$ cannot be solved in integers x, y, z, with $xyz \neq 0$. This theorem has been proved (1979) for all $n < 30{,}000$, but the general theorem is remarkably elusive. Due to the peculiar history of this problem, it has attracted more than its share of mathematical crackpottery and most mathematicians ardently wish that the problem would be settled.

The prime number theorem, which is the subject of this section, has great attractions and mystery and is related to some of the central objects of mathematical analysis. It is also related to what is probably the most famous of the unsolved mathematical problems—the so-called Riemann Hypothesis. It is one of the finest examples of the extraction of order from chaos in the whole of mathematics.

Soon after a child learns to multiply and divide, he notices that some numbers are special. When a number is factored, it is decomposed into its basic constituents—its prime factors. Thus, $6 = 2 \times 3$, $28 = 2 \times 2 \times 7$, $270 = 2 \times 3 \times 3 \times 3 \times 5$ and these decompositions cannot be carried further. The numbers 2, 3, 5, 7, . . . are the prime numbers, numbers that cannot themselves be split into further multiplications. Among the integers, the prime numbers play a role that is analogous to the elements of chemistry.

Let us make a list of the first few prime numbers:

2	3	5	7	11	13	17	19	23	29
31	37	41	43	47	53	59	61	67	71
73	79	83	89	97	101	103	107	109	113 . . .

This list never ends. Euclid already had proved that there are an infinite number of primes. Euclid's proof is easy and elegant and we will give it.

Suppose we have a complete list of all the prime numbers up to a certain prime p_m. Consider the integer $N = (2 \cdot 3 \cdot 5 \cdots p_m) + 1$, formed by adding 1 to the product

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	2	547	1229	1993	2749	3581	4421	5281	6143	7001	7927	8837	9739	10663	11677	12569	13513	14533	15413	16411	17393	18329	19427	20359	21391
2	3	557	1231	1997	2753	3583	4423	5297	6151	7013	7933	8839	9743	10667	11681	12577	13523	14537	15427	16417	17401	18341	19429	20369	21397
3	5	563	1237	1999	2767	3593	4441	5303	6163	7019	7937	8849	9749	10687	11689	12583	13537	14543	15439	16421	17417	18353	19433	20389	21401
4	7	569	1249	2003	2777	3607	4447	5309	6173	7027	7949	8861	9767	10691	11699	12589	13553	14549	15443	16427	17419	18367	19441	20393	21407
5	11	571	1259	2011	2789	3613	4451	5323	6197	7039	7951	8863	9769	10709	11701	12601	13567	14551	15451	16433	17431	18371	19447	20399	21419
6	13	577	1277	2017	2791	3617	4457	5333	6199	7043	7963	8867	9781	10711	11717	12611	13577	14557	15461	16447	17443	18379	19457	20407	21433
7	17	587	1279	2027	2797	3623	4463	5347	6203	7057	7993	8887	9787	10723	11719	12613	13591	14561	15467	16451	17449	18397	19463	20411	21467
8	19	593	1283	2029	2801	3631	4481	5351	6211	7069	8009	8893	9791	10729	11731	12619	13597	14563	15473	16453	17467	18401	19469	20431	21481
9	23	599	1289	2039	2803	3637	4483	5381	6217	7079	8011	8923	9803	10733	11743	12637	13613	14591	15493	16477	17471	18413	19471	20441	21487
10	29	601	1291	2053	2819	3643	4493	5387	6221	7103	8017	8929	9811	10739	11777	12641	13619	14593	15497	16481	17477	18427	19477	20443	21491
11	31	607	1297	2063	2833	3659	4507	5393	6229	7109	8039	8933	9817	10753	11779	12647	13627	14621	15511	16487	17483	18433	19483	20477	21493
12	37	613	1301	2069	2837	3671	4513	5399	6247	7121	8053	8941	9829	10771	11783	12653	13633	14627	15527	16493	17489	18439	19489	20479	21499
13	41	617	1303	2081	2843	3673	4517	5407	6257	7127	8059	8951	9833	10781	11789	12659	13649	14629	15541	16519	17491	18443	19501	20483	21503
14	43	619	1307	2083	2851	3677	4519	5413	6263	7129	8069	8963	9839	10789	11801	12671	13669	14633	15551	16529	17497	18451	19507	20507	21517
15	47	631	1319	2087	2857	3691	4523	5417	6269	7151	8081	8969	9851	10799	11807	12689	13679	14639	15559	16547	17509	18457	19531	20509	21521
16	53	641	1321	2089	2861	3697	4547	5419	6271	7159	8087	8971	9857	10831	11813	12697	13681	14653	15569	16553	17519	18461	19541	20521	21523
17	59	643	1327	2099	2879	3701	4549	5431	6277	7177	8089	8999	9859	10837	11821	12703	13687	14657	15581	16561	17539	18481	19543	20533	21529
18	61	647	1361	2111	2887	3709	4561	5437	6287	7187	8093	9001	9871	10847	11827	12713	13691	14669	15583	16567	17551	18493	19553	20543	21557
19	67	653	1367	2113	2897	3719	4567	5441	6299	7193	8101	9007	9883	10853	11831	12721	13693	14683	15601	16573	17569	18503	19559	20549	21559
20	71	659	1373	2129	2903	3727	4583	5443	6301	7207	8111	9011	9887	10859	11833	12739	13697	14699	15607	16603	17573	18517	19571	20551	21563
21	73	661	1381	2131	2909	3733	4591	5449	6311	7211	8117	9013	9901	10861	11839	12743	13709	14713	15619	16607	17579	18521	19577	20563	21569
22	79	673	1399	2137	2917	3739	4597	5471	6317	7213	8123	9029	9907	10867	11863	12757	13711	14717	15629	16619	17581	18523	19583	20593	21577
23	83	677	1409	2141	2927	3761	4603	5477	6323	7219	8147	9041	9923	10883	11867	12763	13721	14723	15641	16631	17597	18539	19597	20599	21587
24	89	683	1423	2143	2939	3767	4621	5479	6329	7229	8161	9043	9929	10889	11887	12781	13723	14731	15643	16633	17599	18541	19603	20611	21589
25	97	691	1427	2153	2953	3769	4637	5483	6337	7237	8167	9049	9931	10891	11897	12791	13729	14737	15647	16649	17609	18553	19609	20627	21599
26	101	701	1429	2161	2957	3779	4639	5501	6343	7243	8171	9059	9941	10903	11903	12799	13751	14741	15649	16651	17623	18583	19661	20639	21601
27	103	709	1433	2179	2963	3793	4643	5503	6353	7247	8179	9067	9949	10909	11909	12809	13757	14747	15661	16657	17627	18587	19681	20641	21611
28	107	719	1439	2203	2969	3797	4649	5507	6359	7253	8191	9091	9967	10937	11923	12821	13759	14753	15667	16661	17657	18593	19687	20663	21613
29	109	727	1447	2207	2971	3803	4651	5519	6361	7283	8209	9103	9973	10939	11927	12823	13763	14759	15671	16673	17659	18617	19697	20681	21617
30	113	733	1451	2213	2999	3821	4657	5521	6367	7297	8219	9109	10007	10949	11933	12829	13781	14767	15679	16691	17669	18637	19699	20693	21647
31	127	739	1453	2221	3001	3823	4663	5527	6373	7307	8221	9127	10009	10957	11939	12841	13789	14771	15683	16693	17681	18661	19709	20707	21649
32	131	743	1459	2237	3011	3833	4673	5531	6379	7309	8231	9133	10037	10973	11941	12853	13799	14779	15727	16699	17683	18671	19717	20717	21661
33	137	751	1471	2239	3019	3847	4679	5557	6389	7321	8233	9137	10039	10979	11953	12889	13807	14783	15731	16703	17707	18679	19727	20719	21673
34	139	757	1481	2243	3023	3851	4691	5563	6397	7331	8237	9151	10061	10987	11959	12893	13829	14797	15733	16729	17713	18691	19739	20731	21683
35	149	761	1483	2251	3037	3853	4703	5569	6421	7333	8243	9157	10067	10993	11969	12899	13831	14813	15737	16741	17729	18701	19751	20743	21701
36	151	769	1487	2267	3041	3863	4721	5573	6427	7349	8263	9161	10069	11003	11971	12907	13841	14821	15739	16747	17737	18713	19753	20747	21713
37	157	773	1489	2269	3049	3877	4723	5581	6449	7351	8269	9173	10079	11027	11981	12911	13859	14827	15749	16759	17747	18719	19759	20749	21727
38	163	787	1493	2273	3061	3881	4729	5591	6451	7369	8273	9181	10091	11047	11987	12917	13873	14831	15761	16763	17749	18731	19763	20753	21737
39	167	797	1499	2281	3067	3889	4733	5623	6469	7393	8287	9187	10093	11057	12007	12919	13877	14843	15767	16787	17761	18743	19777	20759	21739
40	173	809	1511	2287	3079	3907	4751	5639	6473	7411	8291	9199	10099	11059	12011	12923	13879	14851	15773	16811	17783	18749	19793	20771	21751
41	179	811	1523	2293	3083	3911	4759	5641	6481	7417	8293	9203	10103	11069	12037	12941	13883	14867	15787	16823	17789	18757	19801	20773	21757
42	181	821	1531	2297	3089	3917	4783	5647	6491	7433	8297	9209	10111	11071	12041	12953	13901	14869	15791	16829	17791	18773	19813	20789	21767
43	191	823	1543	2309	3109	3919	4787	5651	6521	7451	8311	9221	10133	11083	12043	12959	13903	14879	15797	16831	17807	18787	19819	20807	21773
44	193	827	1549	2311	3119	3923	4789	5653	6529	7457	8317	9227	10139	11087	12049	12967	13907	14887	15803	16843	17827	18793	19841	20809	21787
45	197	829	1553	2333	3121	3929	4793	5657	6547	7459	8329	9239	10141	11093	12071	12973	13913	14891	15809	16871	17837	18797	19843	20849	21799
46	199	839	1559	2339	3137	3931	4799	5659	6551	7477	8353	9241	10151	11113	12073	12979	13921	14897	15817	16879	17839	18803	19853	20857	21803
47	211	853	1567	2341	3163	3943	4801	5669	6553	7481	8363	9257	10159	11117	12097	12983	13931	14923	15823	16883	17851	18839	19861	20873	21817
48	223	857	1571	2347	3167	3947	4813	5683	6563	7487	8369	9277	10163	11119	12101	13001	13933	14929	15859	16889	17863	18859	19867	20879	21821
49	227	859	1579	2351	3169	3967	4817	5689	6569	7489	8377	9281	10169	11131	12107	13003	13963	14939	15877	16901	17881	18869	19889	20887	21839
50	229	863	1583	2357	3181	3989	4831	5693	6571	7499	8387	9283	10177	11149	12109	13007	13967	14947	15881	16903	17891	18899	19891	20897	21841
51	233	877	1597	2371	3187	4001	4861	5701	6577	7507	8389	9293	10181	11159	12113	13009	13997	14951	15887	16921	17903	18911	19913	20899	21851
52	239	881	1601	2377	3191	4003	4871	5711	6581	7517	8419	9311	10193	11161	12119	13033	13999	14957	15889	16927	17909	18913	19919	20903	21859
53	241	883	1607	2381	3203	4007	4877	5717	6599	7523	8423	9319	10211	11171	12143	13037	14009	14969	15901	16931	17911	18917	19927	20921	21863
54	251	887	1609	2383	3209	4013	4889	5737	6607	7529	8429	9323	10223	11173	12149	13043	14011	14983	15907	16937	17921	18919	19937	20929	21871
55	257	907	1613	2389	3217	4019	4903	5741	6619	7537	8431	9337	10243	11177	12157	13049	14029	15013	15913	16943	17923	18947	19949	20939	21881
56	263	911	1619	2393	3221	4021	4909	5743	6637	7541	8443	9341	10247	11197	12161	13063	14033	15017	15919	16963	17929	18959	19961	20947	21893
57	269	919	1621	2399	3229	4027	4919	5749	6653	7547	8447	9343	10253	11213	12163	13093	14051	15031	15923	16979	17939	18973	19963	20959	21911
58	271	929	1627	2411	3251	4049	4931	5779	6659	7549	8461	9349	10259	11239	12197	13099	14057	15053	15937	16981	17957	18979	19973	20963	21929
59	277	937	1637	2417	3253	4051	4933	5783	6661	7559	8467	9371	10267	11243	12203	13103	14071	15061	15959	16987	17959	19001	19979	20981	21937
60	281	941	1657	2423	3257	4057	4937	5791	6673	7561	8501	9377	10271	11251	12211	13109	14081	15073	15971	16993	17971	19009	19991	20983	21943
61	283	947	1663	2437	3259	4073	4943	5801	6679	7573	8513	9391	10273	11257	12227	13121	14083	15077	15973	17011	17977	19013	19993	21001	21961
62	293	953	1667	2441	3271	4079	4951	5807	6689	7577	8521	9397	10289	11261	12239	13127	14087	15083	15991	17021	17981	19031	19997	21011	21977
63	307	967	1669	2447	3299	4091	4957	5813	6691	7583	8527	9403	10301	11273	12241	13147	14107	15091	16001	17027	17987	19037	20011	21013	21991
64	311	971	1693	2459	3301	4093	4967	5821	6701	7589	8537	9413	10303	11279	12251	13151	14143	15101	16007	17029	17989	19051	20021	21017	21997
65	313	977	1697	2467	3307	4099	4969	5827	6703	7591	8539	9419	10313	11287	12253	13159	14149	15107	16033	17033	18013	19069	20023	21019	22003
66	317	983	1699	2473	3313	4111	4973	5839	6709	7603	8543	9421	10321	11299	12263	13163	14153	15121	16057	17041	18041	19073	20029	21023	22013
67	331	991	1709	2477	3319	4127	4987	5843	6719	7607	8563	9431	10331	11311	12269	13171	14159	15131	16061	17047	18043	19079	20047	21031	22027
68	337	997	1721	2503	3323	4129	4993	5849	6733	7621	8573	9433	10333	11317	12277	13177	14173	15137	16063	17053	18047	19081	20051	21059	22031
69	347	1009	1723	2521	3329	4133	4999	5851	6737	7639	8581	9437	10337	11321	12281	13183	14177	15139	16067	17077	18049	19087	20063	21061	22037
70	349	1013	1733	2531	3331	4139	5003	5857	6761	7643	8597	9439	10343	11329	12289	13187	14197	15149	16069	17093	18059	19121	20071	21067	22039
71	353	1019	1741	2539	3343	4153	5009	5861	6763	7649	8599	9461	10357	11351	12301	13217	14207	15161	16073	17099	18061	19139	20089	21089	22051
72	359	1021	1747	2543	3347	4157	5011	5867	6779	7669	8609	9463	10369	11353	12323	13219	14221	15173	16087	17107	18077	19141	20101	21101	22063
73	367	1031	1753	2549	3359	4159	5021	5869	6781	7673	8623	9467	10391	11369	12329	13229	14243	15187	16091	17117	18089	19157	20107	21107	22067
74	373	1033	1759	2551	3361	4177	5023	5879	6791	7681	8627	9473	10399	11383	12343	13241	14249	15193	16097	17123	18097	19163	20113	21121	22073
75	379	1039	1777	2557	3371	4201	5039	5881	6793	7687	8629	9479	10427	11393	12347	13249	14251	15199	16103	17137	18119	19181	20117	21139	22079
76	383	1049	1783	2579	3373	4211	5051	5897	6803	7691	8641	9491	10429	11399	12373	13259	14281	15217	16111	17159	18121	19183	20123	21143	22091
77	389	1051	1787	2591	3389	4217	5059	5903	6823	7699	8647	9497	10433	11411	12377	13267	14293	15227	16127	17167	18127	19207	20129	21149	22093
78	397	1061	1789	2593	3391	4219	5077	5923	6827	7703	8663	9511	10453	11423	12379	13291	14303	15233	16139	17183	18131	19211	20143	21157	22109
79	401	1063	1801	2609	3407	4229	5081	5927	6829	7717	8669	9521	10457	11437	12391	13297	14321	15241	16141	17189	18133	19213	20147	21163	22111
80	409	1069	1811	2617	3413	4231	5087	5939	6833	7723	8677	9533	10459	11443	12401	13309	14323	15259	16183	17191	18143	19219	20149	21169	22123
81	419	1087	1823	2621	3433	4241	5099	5953	6841	7727	8681	9539	10463	11447	12409	13313	14327	15263	16187	17203	18149	19231	20161	21179	22129
82	421	1091	1831	2633	3449	4243	5101	5981	6857	7741	8689	9547	10477	11467	12413	13327	14341	15269	16189	17207	18169	19237	20173	21187	22133
83	431	1093	1847	2647	3457	4253	5107	5987	6863	7753	8693	9551	10487	11471	12421	13331	14347	15271	16193	17209	18181	19249	20177	21191	22147
84	433	1097	1861	2657	3461	4259	5113	6007	6869	7757	8699	9587	10499	11483	12433	13337	14369	15277	16217	17231	18191	19259	20183	21193	22153
85	439	1103	1867	2659	3463	4261	5119	6011	6871	7759	8707	9601	10501	11489	12437	13339	14387	15287	16223	17239	18199	19267	20201	21211	22157
86	443	1109	1871	2663	3467	4271	5147	6029	6883	7789	8713	9613	10513	11491	12451	13367	14389	15289	16229	17257	18211	19273	20219	21221	22159
87	449	1117	1873	2671	3469	4273	5153	6037	6899	7793	8719	9619	10529	11497	12457	13381	14401	15299	16231	17291	18217	19289	20231	21227	22171
88	457	1123	1877	2677	3491	4283	5167	6043	6907	7817	8731	9623	10531	11503	12473	13397	14407	15307	16249	17293	18223	19301	20233	21247	22189
89	461	1129	1879	2683	3499	4289	5171	6047	6911	7823	8737	9629	10559	11519	12479	13399	14411	15313	16253	17299	18229	19309	20249	21269	22193
90	463	1151	1889	2687	3511	4297	5179	6053	6917	7829	8741	9631	10567	11527	12487	13411	14419	15319	16267	17317	18233	19319	20261	21277	22229
91	467	1153	1901	2689	3517	4327	5189	6067	6947	7841	8747	9643	10589	11549	12491	13417	14423	15329	16273	17321	18251	19333	20269	21283	22247
92	479	1163	1907	2693	3527	4337	5197	6073	6949	7853	8753	9649	10597	11551	12497	13421	14431	15331	16301	17327	18253	19373	20287	21313	22259
93	487	1171	1913	2699	3529	4339	5209	6079	6959	7867	8761	9661	10601	11579	12503	13441	14437	15349	16319	17333	18257	19379	20297	21317	22271
94	491	1181	1931	2707	3533	4349	5227	6089	6961	7873	8779	9677	10607	11587	12511	13451	14447	15359	16333	17341	18269	19381	20323	21319	22273
95	499	1187	1933	2711	3539	4357	5231	6091	6967	7877	8783	9679	10613	11593	12517	13457	14449	15361	16339	17351	18287	19387	20327	21323	22277
96	503	1193	1949	2713	3541	4363	5233	6101	6971	7879	8803	9689	10627	11597	12527	13463	14461	15373	16349	17359	18289	19391	20333	21341	22279
97	509	1201	1951	2719	3547	4373	5237	6113	6977	7883	8807	9697	10631	11617	12539	13469	14479	15377	16361	17377	18301	19403	20341	21347	22283
98	521	1213	1973	2729	3557	4391	5261	6121	6983	7901	8819	9719	10639	11621	12541	13477	14489	15383	16363	17383	18307	19417	20347	21377	22291
99	523	1217	1979	2731	3559	4397	5273	6131	6991	7907	8821	9721	10651	11633	12547	13487	14503	15391	16369	17387	18311	19421	20353	21379	22303
100	541	1223	1987	2741	3571	4409	5279	6133	6997	7919	8831	9733	10657	11657	12553	13499	14519	15401	16381	17389	18313	19423	20357	21383	22307

Table of the First 2500 Prime Numbers

From D. N. Lehmer, List of prime numbers from 1 to 10,006,721, Carnegie Institution of Washington. Publication No. 165, Washington, D.C., 1914.

of all the primes up to p_m. Now N is larger than p_m (for it is certainly more than twice its size). When N is divided by 2 it goes $3 \cdot 5 \cdot \cdots \cdot p_m$ times and leaves a remainder 1. When it is divided by 3, it goes $2 \cdot 5 \cdot \cdots \cdot p_m$ times and leaves a remainder 1. Similarly, it leaves a remainder of 1 when divided by any of the primes $2, 3, 5, \ldots, p_m$.

Now N is either a prime number or it isn't. If it is a prime number, it is a prime number greater than p_m. If it isn't a prime number, it may be factored into prime numbers. But none of its prime factors can be $2, 3, 5, \ldots, p_m$ as we just saw. Therefore there is a prime number greater than p_m.

The logical argument (actually, the dilemma, which forces one to the same conclusion whichever path one is compelled to take) tells us that the list of primes never ends.

The second feature of the list of primes that strikes one is the absence of any noticeable pattern or regularity. Of course all the prime numbers except 2 are odd, so the gap between any two successive primes has to be an even number. But there seems to be no rhyme or reason as to which even number it happens to be.

There are nine prime numbers between 9,999,900 and 10,000,000:

9,999,901	9,999,907	9,999,929	9,999,931
9,999,937	9,999,943	9,999,971	9,999,973
9,999,991.			

But among the next hundred integers, from 10,000,000 to 10,000,100, there are only two:

10,000,019 and 10,000,079.

"Upon looking at these numbers, one has the feeling of being in the presence of one of the inexplicable secrets of creation," writes Don Zagier in an outburst of modern number mysticism.

What is known about primes and what is not known or conjectural would fill a large book. Here are some samples. The largest known prime in 1979 was $2^{21,701} - 1$. There is a prime between n and $2n$ for every integer $n > 1$. Is there a prime between n^2 and $(n + 1)^2$ for every $n > 0$? No one knows. Are there an infinity of primes of the form $n^2 + 1$ where n is an integer? No one knows. There are runs of integers of arbitrary length which are free of primes. No polynomial with integer coefficients can take on only prime values at the integers. There is an irrational number A such that $[A^{3^n}]$ takes on only prime values as $n = 0, 1, 2, \ldots$. (Here the notation $[x]$ means the greatest integer $\leq x$.) Is every even number the sum of two odd primes? No one knows; this is the notorious Goldbach conjecture. Are there an infinite number of prime pairs, such as 11;13 or 17;19 or 10,006,427;10,006,429 which differ by 2? This is the problem of the twin primes, and no one knows the answer though most mathematicians are convinced that the statement is very likely to be true.

Some order begins to emerge from this chaos when the primes are considered not in their individuality but in the aggregate; one considers the social statistics of the primes and not the eccentricities of the individuals. One first makes a large tabulation of primes. This is difficult and tedious with pencil and paper, but with a modern computer it is easy. Then one counts them to see how many there are up to a given point. The function $\pi(n)$ is defined as the number of primes less than or equal to the number n. The function $\pi(n)$ measures the distribution of the prime numbers. Having obtained it, it is only natural to compute the ratio $n/\pi(n)$ which tells us what fraction of the numbers up to a given point are primes. (Actually, it is the reciprocal of this fraction.) Here is the result of a recent computation.

n	π(n)	n/π(n)
10	4	2.5
100	25	4.0
1000	168	6.0
10,000	1,229	8.1
100,000	9,592	10.4
1,000,000	78,498	12.7
10,000,000	664,579	15.0
100,000,000	5,761,455	17.4
1,000,000,000	50,847,534	19.7
10,000,000,000	455,052,512	22.0

Notice that as one moves from one power of 10 to the next, the ratio $n/\pi(n)$ increases by roughly 2.3. (For example, 22.0 − 19.7 = 2.3.) At this point, any mathematician worth his salt thinks of $\log_e 10$ (=2.30258 . . .) and on the basis of this evidence, it is easy to formulate the conjecture that $\pi(n)$ is approximately equal to $\frac{n}{\log n}$. The more formal statement that

$$\lim_{n\to\infty} \pi(n)/(n/\log n) = 1$$

Carl Friedrich Gauss
1777–1855

is the famous prime number theorem. The discovery of the theorem can be traced as far back as Gauss, at age fifteen (around 1792), but the rigorous mathematical proof dates from 1896 and the independent work of C. de la Vallée Poussin and Jacques Hadamard. Here is order extracted from confusion, providing a moral lesson on how individual eccentricities can exist side by side with law and order.

Jacques Hadamard
1865–1963

While the expression $n/\log n$ is a fairly simple approximation for $\pi(n)$, it is not terribly close, and mathematicians have been interested in improving it. Of course, one does this at the price of complicating the approximant. One of the most satisfactory approximants to $\pi(n)$ is the function

$$R(n) = 1 + \sum_{k=1}^{\infty} \frac{1}{k\zeta(k+1)} \frac{(\log n)^k}{k!}$$

where $\zeta(z)$ designates the celebrated Riemann zeta function: $\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \ldots$. The accompanying table shows what a remarkably good approximation $R(n)$ is to $\pi(n)$:

	$\pi(n)$	**R**(n)
100,000,000	5,761,455	5,761,552
200,000,000	11,078,937	11,079,090
300,000,000	16,252,325	16,252,355
400,000,000	21,336,326	21,336,185
500,000,000	26,355,867	26,355,517
600,000,000	31,324,703	31,324,622
700,000,000	36,252,931	36,252,719
800,000,000	41,146,179	41,146,248
900,000,000	46,009,215	46,009,949
1,000,000,000	50,847,534	50,847,455

Let us turn, finally, to the question of twin prime pairs. It is thought that there are an infinite number of such pairs, though this is still an open question.

Why do we believe it is true, even though there is no proof? First of all, there is numerical evidence; we find more prime pairs whenever we look for them; there does not seem to be a region of the natural number system so remote that it lies beyond the largest prime pair. But more than that, we have an idea *how many* prime pairs there are. We can get this idea by noticing that the occurrence of prime pairs in a table of prime numbers seem to be unpredictable or *random.* This suggests the conjecture that the chance of two numbers n and $n + 2$, both being prime, acts like the chance of getting a head on two successive tosses of a coin. If two successive random experiments are independent, the chance of success on both is the product of the chances of success on either; for example, if one coin has probability $\frac{1}{2}$ of coming up heads, two coins have probability $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ of coming up a pair of heads.

Now the prime number theorem, which *has* been proved, says that if n is a large number, and we choose a number x at random between 0 and n, the chance that x is prime will be "about" $\frac{1}{\log n}$. The bigger n is, the better is the approximation given by $\frac{1}{\log n}$ to the proportion of primes in the numbers up to n.

If we trust our feeling that the occurrence of twin primes is like two coins coming up heads, then the chance that both x and $x + 2$ are prime would be about $\frac{1}{(\log n)^2}$. In other words, there would be about $\frac{n}{(\log n)^2}$ prime pairs to be found between 0 and n. This fraction approaches infinity as n goes to infinity, so this would provide a quantitative version of the prime pair conjecture.

For reasons involving the dependence of $x + 2$ being prime on the supposition that x is already prime, one should modify the estimate $\frac{n}{(\log n)^2}$ to $\frac{(1.32032..)n}{(\log n)^2}$.

Appended is a comparison between what has been found and what is predicted by this simple formula. The agreement is remarkably good, but the final Q.E.D. is yet to be written.

Interval	Prime twins expected	Prime twins found
100,000,000–100,150,000	584	601
1,000,000,000–1,000,150,000	461	466
10,000,000,000–10,000,150,000	374	389
100,000,000,000–100,000,150,000	309	276
1,000,000,000,000–1,000,000,150,000	259	276
10,000,000,000,000–10,000,000,150,000	221	208
100,000,000,000,000–100,000,000,150,000	191	186
1,000,000,000,000,000–1,000,000,000,150,000	166	161

Further Readings. See Bibliography

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The Mathematical Experience

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