

ED1110 VEKTORANALYS

SUMMARY

HT 2021

Warning: I have tried to be extremely concise. So this summary might not contain all the topics that you need to learn. It might contain minor imprecisions and typos. If you find any of them, please contact me.

1. GRADIENT
2. LINE INTEGRAL
3. FLUX INTEGRAL
4. DIVERGENCE AND CURL
5. GAUSS' THEOREM and STOKES' THEOREM
6. CURVILINEAR COORDINATES
7. NABLA OPERATOR AND NABLARÄKNING
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10. IMPORTANT VECTOR FIELDS
11. LAPLACE AND POISSON EQUATIONS

1. GRADIENT

A **scalar field** associates a **real number** $\phi(x,y,z)$ to **each point** (x,y,z) of the space.

A **vector field** associates a **vector** $\vec{A}(x,y,z)$ to **each point** (x,y,z) of the space.

A level surface is a **surface** on which the scalar field **is constant**: $\phi(x,y,z)=c$

The **gradient** of a scalar ϕ is the vector defined by:

$$\text{grad}\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} \right) \quad (\text{in cartesian})$$

The **increase of a scalar field in direction** $\hat{e}$ is
the component of the gradient in that direction:

Directional derivative

$$\frac{d\phi}{ds} = \text{grad}\phi \cdot \hat{e}$$

THEOREM: The **direction of** $\text{grad}\phi$ is the **direction of the maximum increase** of ϕ

The maximum increase of ϕ per unit length is $|\text{grad}\phi|$

THEOREM: The gradient in the point P is zero if ϕ has a maximum or a minimum in P

THEOREM: $\text{grad}\phi$ is orthogonal to the level surfaces of ϕ .

2. LINE INTEGRAL

The line integral of the vector field $\vec{F}$ on the path L parameterized by the equation $\vec{r}=\vec{r}(u)$ is:

$$\int_L \vec{F}(\vec{r}) \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(u)) \cdot \frac{d\vec{r}}{du} du$$

THEOREM: The line integral along $-L$ is the opposite of the line integral along L

$$\int_{-L} \vec{F}(\vec{r}) \cdot d\vec{r} = - \int_L \vec{F}(\vec{r}) \cdot d\vec{r}$$

The line of integral along a closed path is called circulation (*cirkulationen*)

$$\oint_C \vec{A}(\vec{r}) \cdot d\vec{r}$$

THEOREM: The **circulation is zero** if and only if for all points P and Q the **line integral is independent from the integration path between P and Q**.

Such a field is called **conservative**.

THEOREM: If $\vec{A} = \text{grad} \phi$ then : $\int_P^Q \vec{A}(\vec{r}) \cdot d\vec{r} = \phi(Q) - \phi(P)$

3. FLUX INTEGRAL

The flux integral of the vector field $\vec{A}$ on the surface S parameterized by $\vec{r}=\vec{r}(u,v)$ is:

$$\iint_S \vec{A} \cdot d\vec{S} = \int_{a_1}^{a_2} \int_{b_1}^{b_2} \vec{A}(\vec{r}(u,v)) \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) du dv$$

The infinitesimal surface element $d\vec{S}$ is:

$$d\vec{S} = \hat{n} dS = \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) du dv$$

4. DIVERGENCE AND CURL

The **divergence** of a vector $\vec{A}$ is a **scalar** defined by:

It is a measure of **how much the field diverges (or converges) from (to) a point**.

$$\text{div} \vec{A} \equiv \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

The **curl** of a vector $\vec{A}$ is a vector defined by:

The curl is a measure of how much the direction of a vector field changes in space, i.e. how much the field “rotates”.

The **direction is the axis of rotation**. The **amplitude is the amplitude of rotation**.

$$\text{rot} \vec{A} \equiv \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

THEOREM: A vector field $\vec{A}$ has a scalar potential ϕ if and only if $\text{rot} \vec{A} = 0$ (virvålfria fält)

THEOREM: $\vec{B}$ has a vector potential $\vec{A}$, $\vec{B} = \text{rot} \vec{A} \Leftrightarrow \text{div} \vec{B} = 0$ (solenoidal fält)

5. GAUSS' THEOREM and STOKES' THEOREM

GAUSS' THEOREM

$$\oiint_S \vec{A} \cdot d\vec{S} = \iiint_V \text{div} \vec{A} dV$$

where S is a closed surface that forms the boundary of the volume V and $\vec{A}$ is a continuously differentiable vector field on V .

STOKES' THEOREM

$$\oint_L \vec{A} \cdot d\vec{r} = \iint_S \text{rot} \vec{A} \cdot d\vec{S}$$

where $\vec{A}$ is a continuously differentiable vector field on S , L is a closed curve and S is a surface with boundary on L .

6. CURVILINEAR COORDINATES

The basis $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ of an orthogonal curvilinear coordinate system (u_1, u_2, u_3) are defined as:

$$\hat{e}_i = \frac{1}{h_i} \frac{\partial \vec{r}}{\partial u_i} \quad \text{with scale factor} \quad h_i = \left| \frac{\partial \vec{r}}{\partial u_i} \right|$$

In a curvilinear coordinate system, gradient, curl and divergence are:

$$\text{grad} \phi = \sum_i \frac{1}{h_i} \frac{\partial \phi}{\partial u_i} \hat{e}_i$$

$$\text{rot} \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

$$\text{div} \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (A_1 h_2 h_3) + \frac{\partial}{\partial u_2} (A_2 h_3 h_1) + \frac{\partial}{\partial u_3} (A_3 h_1 h_2) \right]$$

See the formula sheet for applications to cylindrical and spherical coordinate systems

7. NABLA OPERATOR AND NABLARÄKNING

An **operator** T is a law that to each function f in the function class D_t associates a function $T(f)$.

An operator T is **linear** if $T(af+bg)=aT(f)+bT(g)$, where f and g are two functions

The **nabla** operator is:

$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

$$\begin{aligned} \text{grad} \phi &= \nabla \phi \\ \text{div} \vec{A} &= \nabla \cdot \vec{A} \\ \text{rot} \vec{A} &= \nabla \times \vec{A} \end{aligned}$$

To calculate an expression that contains nabla:

STEP 1 **Rewrite the expression as a sum** with N terms, where N is the number of fields in the expression. Every term will be identical to the original expression, but **the i -th field in the i -th term will have a dot**.

$$\nabla \cdot (\phi, \vec{A}, \psi, \vec{B}, \dots) = \nabla \cdot (\phi, \vec{A}, \psi, \vec{B}, \dots) + \nabla \cdot (\phi, \vec{A}, \psi, \vec{B}, \dots) + \nabla \cdot (\phi, \vec{A}, \psi, \vec{B}, \dots) + \nabla \cdot (\phi, \vec{A}, \psi, \vec{B}, \dots) + \dots$$

STEP 2 **The nabla can now be formally considered as a vector**. Each term will be rewritten **using vector algebra rules**.

STEP 3 In each rewritten term, **ONLY the field with the “dot” must appear after the nabla**. Finally, you can remove the “dot”.

8. INDEXRÄKNING

Notations: substitute x, y, z with suffices $1, 2, 3$. Examples: $A_x = A_1$ $\hat{e}_y = \hat{e}_2$ $\frac{\partial \phi}{\partial y} = \partial_2 \phi$ $\frac{\partial A_x}{\partial y} = \partial_2 A_1$

Summation convention: whenever a suffix is repeated in a single term, summation from 1 to 3 is understood. Example: $\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z = a_i b_i$

Kronecker delta $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$ some properties: $l_{jm} \delta_{km} = l_{jk}$ $\delta_{ii} = 3$

Alternating tensor

$$\varepsilon_{ijk} = \hat{e}_i \cdot (\hat{e}_j \times \hat{e}_k) = \begin{cases} 0 & \text{if any of } i, j, k \text{ are equal} \\ +1 & \text{if } (i, j, k) = (1, 2, 3) \text{ or } (2, 3, 1) \text{ or } (3, 1, 2) \quad (\text{even permutation}) \\ -1 & \text{if } (i, j, k) = (1, 3, 2) \text{ or } (2, 1, 3) \text{ or } (3, 2, 1) \quad (\text{odd permutation}) \end{cases}$$

The alternating tensor is useful to write the cross product in tensor notation:

$$(\vec{a} \times \vec{b})_i = \varepsilon_{ijk} a_j b_k$$

The nabla calculation can be performed using the tensor notation, the Kronecker delta and the alternating tensor. This procedure is called Index calculation (**indexräkning**).

Gradient, divergence and curl in index notation:

$$(\nabla \phi)_i = \phi_{,i}$$

$$\nabla \cdot \vec{A} = A_{i,i}$$

$$(\nabla \times \vec{A})_i = \varepsilon_{ijk} A_{k,j}$$

9. INTEGRAL THEOREMS

Integral theorems are a generalization of Gauss and Stokes' theorem

Generalized Gauss theorem:

$$\oiint_S d\vec{S}(\dots) = \iiint_V dV \nabla(\dots)$$

where (...) can be substituted with:

$$\phi, \cdot \vec{A}, \times \vec{A}$$

Generalized Stokes theorem:

$$\oint_L d\vec{r}(\dots) = \iint_S (d\vec{S} \times \nabla)(\dots)$$

10. IMPORTANT VECTOR FIELDS

POINT SOURCE

The **vector field generated by a point source** is:

$$\vec{A}(\vec{r}) = \frac{S}{r^2} \hat{e}_r$$

The **flux** of the field generated by a **point source** is:

$$\oiint_S \frac{S}{r^2} \hat{e}_r \cdot d\vec{S} = \begin{cases} 0 & \text{If the source is outside } V \\ 4\pi S & \text{If the source is inside } V \end{cases}$$

The **potential from a point source** is:

$$\phi = -\frac{S}{r}$$

DIPOLE

The potential and the electric field generated by a dipole are:

$$\phi(\vec{r}) = \frac{\vec{p} \cdot \vec{r}}{r^3}$$

$$\vec{E}(\vec{r}) = -\frac{\vec{p}}{r^3} + \frac{3(\vec{p} \cdot \vec{r})\vec{r}}{r^5}$$

where $\vec{p}$ is the dipole moment

VIRVELTRÅDEN

$$\oint_L \frac{k}{\rho} \hat{e}_\phi \cdot d\vec{r} = 2\pi kN$$

Where N is number of turns of L around the z -axis

11. LAPLACE AND POISSON EQUATIONS

Laplace's equation $\nabla^2 \phi = 0$

Poisson's equation $\nabla^2 \phi = \rho$

THEOREM: If ϕ has continuous second derivatives in the volume V and if $\phi=0$ on the surface S that encloses V then the solution to the Laplace equation $\nabla^2 \phi = 0$ is $\phi(x,y,z)=0$ everywhere in V

$$\begin{array}{ll} \text{Dirichlet's} & \nabla^2 \phi = \rho \\ \text{boundary condition} & \phi = \sigma \text{ on } S \end{array}$$

$$\begin{array}{ll} \text{Neumann's} & \nabla^2 \phi = \rho \\ \text{Boundary condition} & \frac{\partial \phi}{\partial n} = \hat{n} \cdot \nabla \phi = \gamma \text{ on } S \end{array}$$

THEOREM: The Poisson's equation $\nabla^2 \phi = \rho$ in the volume V with boundary condition $\phi=\sigma$ on the surface S (boundary of V) has only one solution.

THEOREM: If ϕ_s is a solution to the Poisson's equation $\nabla^2 \phi = \rho$ in V with boundary condition $\hat{n} \cdot \nabla \phi = \gamma$, then also $\phi_s + c$ is a solution, with c an arbitrary constant.

Solutions of the Laplace's equation $\nabla^2 \phi = 0$ in simple cases:

$$\text{PLANAR SYMMETRY} \quad \phi = \phi(x) \Rightarrow \frac{d^2 \phi(x)}{dx^2} = 0 \Rightarrow \phi(x) = ax + b$$

$$\text{CYLINDRICAL SYMMETRY} \quad \phi = \phi(\rho) \Rightarrow \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{d\phi(\rho)}{d\rho} \right) = 0 \Rightarrow \phi(\rho) = a \ln \rho + b$$

$$\text{SPHERICAL SYMMETRY} \quad \phi = \phi(r) \Rightarrow \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi(r)}{dr} \right) = 0 \Rightarrow \phi(r) = -\frac{a}{r} + b$$