VEKTORANALYS

HT 2021

CELTE / CENMI

ED1110

NABLAOPERATOR och NABLARÄKNING, INTEGRALSATSER, TENSORER och INDEXRÄKNING

Kapitel 11, 12, 14
Kapitel 15

version: 26-sept-2021

THIS WEEK

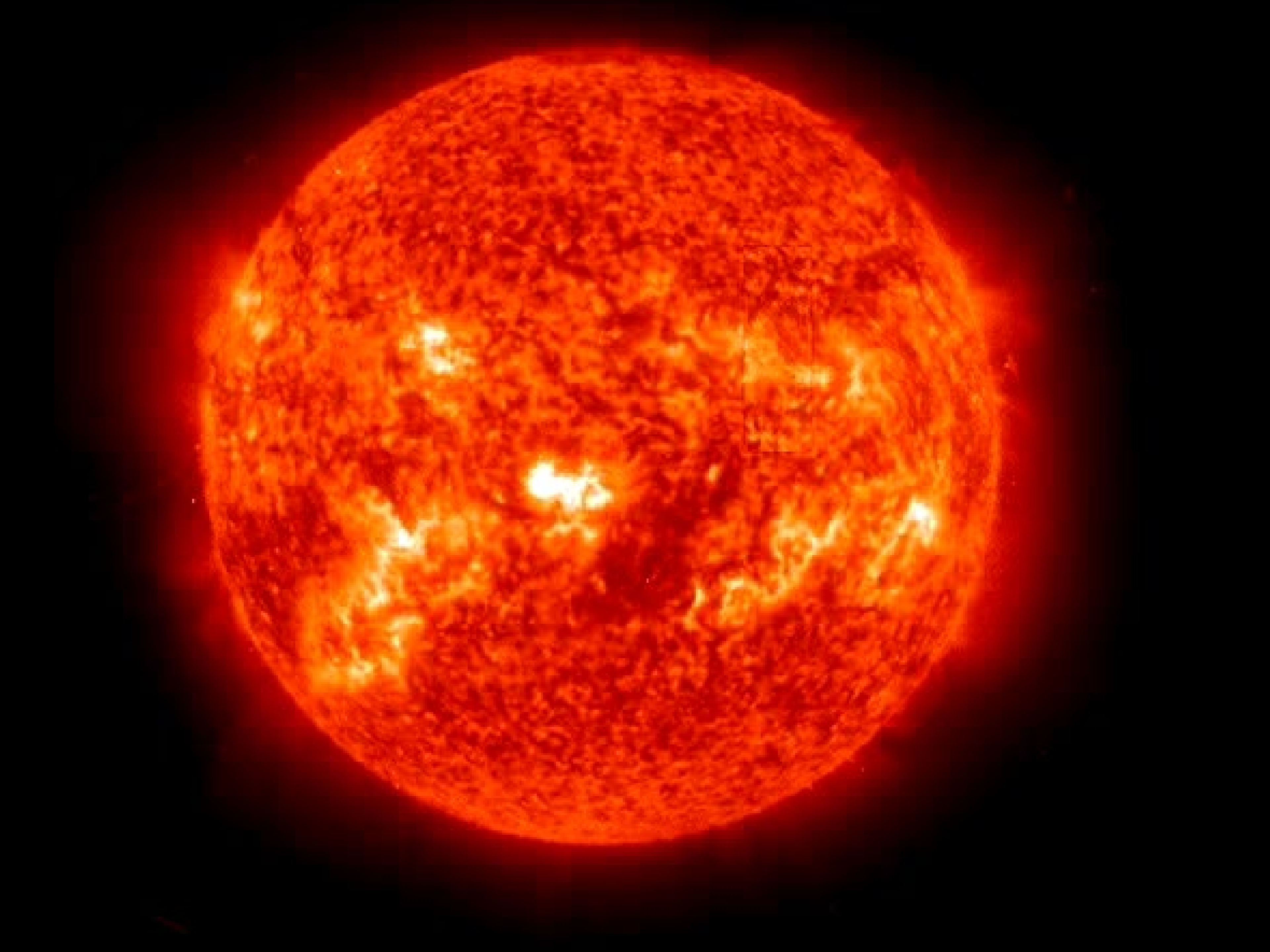
- Nabla
  - Grad div and rot using nabla (chapter 11)
  - laplacian (chapter 14)
- ”Nabla räkning” (chapter 11)
- Example of application of nabla räkning: (chapter 14)
  - from Maxwell’s equations to electromagnetic waves

- ”Integralsaster” (chapter 15)

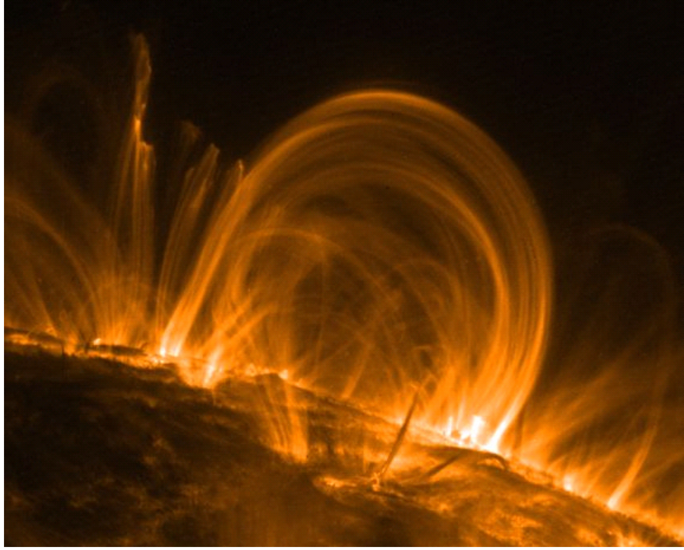
- Indexräkning: (chapter 12)
  - application to vector identities
  - application to nabla identities
- Tensors (not necessary to pass the course) (chapter 13)

Connections with previous and next topics

- Nabla and nabläräkning: connection to gradient, divergence and curl.
- It will help to simplify expressions that contain sevral div, grad and rot (expressions that are often present in electromagentic theory)
- Integralsatser: connection with Gauss' and Stokes' theorems (integralsatser are a generalization of them)



NUCLEAR FUSION



The sun is composed mainly of hydrogen (74%)
and helium (25%)

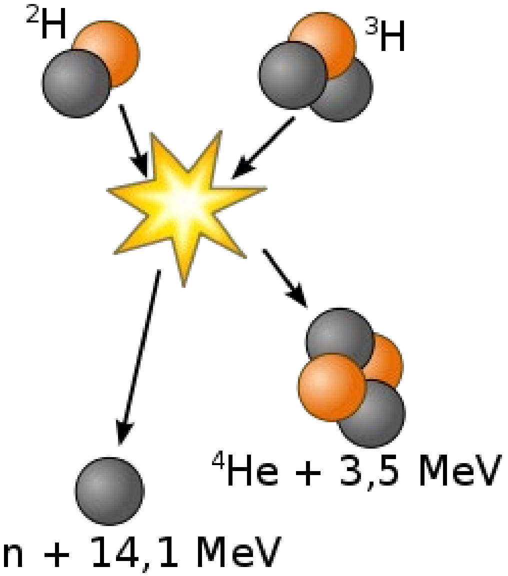
The temperature is so high *(6000K on the surface, 15MK in the core)* that the atoms are ionized:

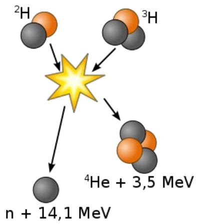
- the sun is basically composed of a "ionized gas" made of electrons and protons
- this kind of "ionized gas" is the fourth state of matter (solid, liquid, gas and): **plasma**

What happens in the sun core?
Protons fuse together and produce helium and energy.
(the actual chain of reactions is more complicated)

On Earth, scientists are trying to use this principle to build a fusion reactor using the reaction:

$^2H + ^3H \rightarrow ^4He + n +$ **energy**





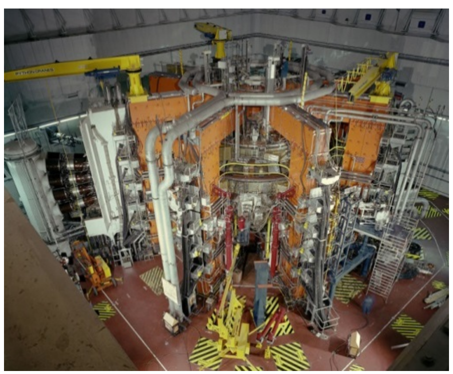
FUSION EXPERIMENTS

$$^{2}H+^{3}H\rightarrow ^{4}He+n+\textbf{energy}$$

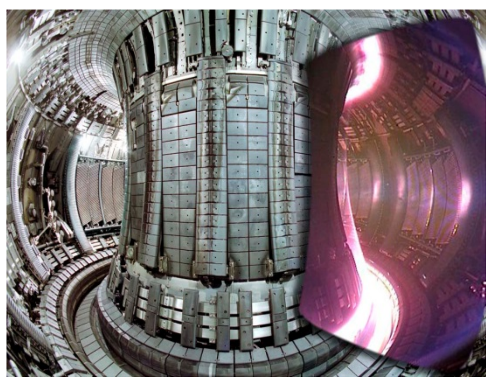
Can we use this method to obtain energy, here on the earth?
Physicists and engineers are working (also at KTH) on it…

The JET experiment
(located near Oxford)
can produce plasmas for ≈20-30sec with
max temperature 50-100 million K
https://www.euro-fusion.org/

At the Division of Fusion Plasma Physics in KTH we reach 5 million K



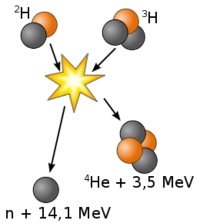
Outer view of JET



Inner view of the plasma chamber in JET
(chamber height and width: 2.1m x 1.25m)

Outer view of EXTRAP T2R at KTH
(chamber height and width: 0.2m x 0.2m)

For more info visit the Division of Fusion Plasma Physics at KTH
or visit the website https://www.kth.se/ee/fpp

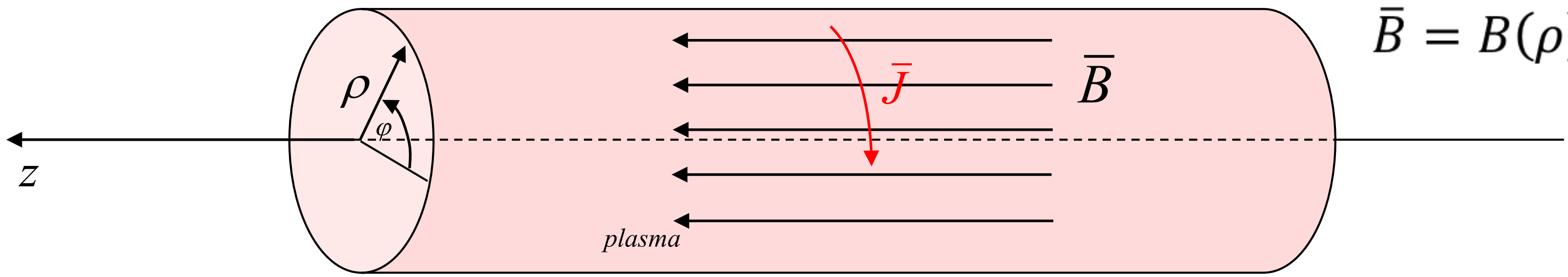


TARGET PROBLEM

In the plasma there are many particles (10^{19}, 10^{20} per m^3), strong magnetic and electric fields and electric currents. How can we describe the behaviour of the plasma?

Magnetohydrodynamics (MHD)

Simple example: **THE THETA PINCH**



$$\bar{B} = B(\rho)\hat{e}_z$$

When the plasma is in equilibrium, the MHD equations can be simplified to:

$$\begin{cases} grad\ p = \bar{j} \times \bar{B} \\ rot\bar{B} = \mu_0 \bar{j} \end{cases} \quad \Rightarrow \quad grad\ p = \frac{1}{\mu_0}\left(rot\bar{B}\right) \times \bar{B}$$

And then?
How to continue?

p is the pressure
$\bar{j}$ is the current density

We need to introduce:

- **Operators**
- **Nabla**

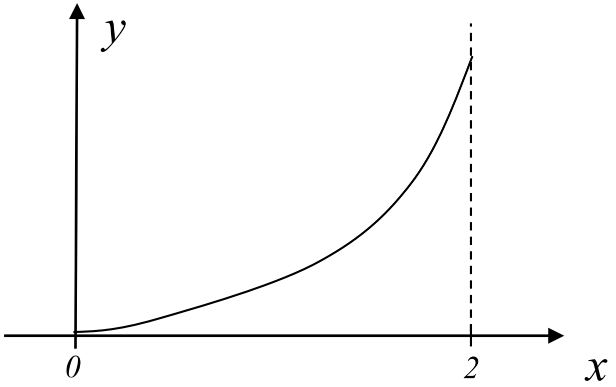
OPERATOR

What is a function?

A function is a law defined in a domain X that to each element x in X associates one and only one element y in Y.

Example: $X=[0,2]$

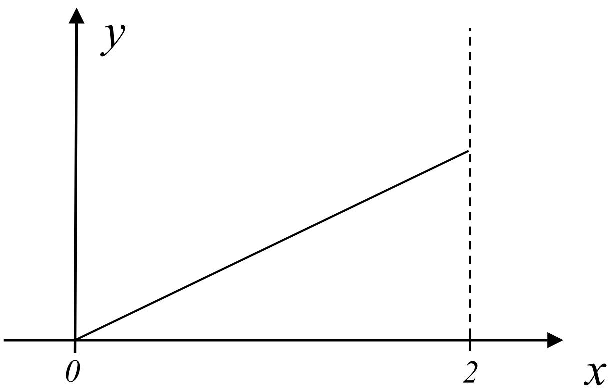
$f(x)=x^2$



The slope of $f(x)$ is its derivative:

$$g(x)=\frac{df(x)}{dx}$$

$g(x)$ is still a function.



So the derivative is **a rule that associates a function to another function**.

The derivative is an example of **operator**

OPERATOR

DEFINITION

An **operator** T is a law that to each function f in the function class D_t associates a function $T(f)$.

DEFINITION

An operator T is **linear** if ***$T(af+bg)=aT(f)+bT(g)$***, where f and g are functions belonging to D_t and a, b constants

EXAMPLE: $T = \frac{d}{dx}$ *is it linear?* YES

where:
f,g are two functions of x
a,b are two constants

$$T(af+bg) = \frac{d(af+bg)}{dx} = a\frac{df}{dx} + b\frac{dg}{dx} = aT(f) + bT(g)$$

SUM AND PRODUCT OF OPERATORS

Sum of two operators $(T+U)(f) = T(f) + U(f)$

Product of two operators $(TU)(f) = T(U(f))$

NABLA

Gradient, divergence and curl have something in common:

$$grad\,\phi \equiv \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)$$

$$grad\,\phi = \nabla \phi$$

$$div\,\overline{A} \equiv \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$div\,\overline{A} = \nabla \cdot \overline{A}$$

$$rot\,\overline{A} \equiv \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$rot\,\overline{A} = \nabla \times \overline{A}$$

$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$ is common to all three definitions

$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

This operator is called
NABLA

THE SCALAR LAPLACIAN, THE VECTOR LAPLACIAN and more

- The divergence of the gradient is called laplacian or **Laplace operator**

$\nabla \cdot \nabla \phi = \nabla^2 \phi$ is the scalar Laplacian of the scalar field ϕ. *Sometimes written as:* $\Delta\phi$

In a Cartesian coordinate system: $\nabla^2 = \nabla \cdot \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)$

$$\boxed{\nabla^2 \phi = \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right)}$$

- In a Cartesian coordinate system, the **vector laplacian** is defined as

$$\boxed{\nabla^2 \bar{A} = \nabla^2 A_x \hat{e}_x + \nabla^2 A_y \hat{e}_y + \nabla^2 A_z \hat{e}_z}$$

- The nabla can be used to define new operators like: $\bar{A} \cdot \nabla$ or $\bar{A} \times \nabla$

Example: $\bar{A} \cdot \nabla = \left(A_x, A_y, A_z \right) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \left(A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z} \right)$

so: $$\boxed{\left(\bar{A} \cdot \nabla \right) \bar{B} = \left(A_x \frac{\partial \bar{B}}{\partial x} + A_y \frac{\partial \bar{B}}{\partial y} + A_z \frac{\partial \bar{B}}{\partial z} \right)}$$

EXERCISE: calculate $\left(\bar{a} \cdot \nabla \right) \bar{r}$ *where* $\bar{a}$ *is constant*

Note that: $\left(\bar{A} \cdot \nabla \right) \bar{B} \neq \bar{A} \left(\nabla \cdot \bar{B} \right)$

EXERCISE: calculate $\bar{a} \left(\nabla \cdot \bar{r} \right)$

IDENTITIES

ϕ and ψ : scalar fields

$\overline{A}$ and $\overline{B}$: vector fields

$$\nabla(\phi\psi) = (\nabla\phi)\psi + \phi(\nabla\psi)$$ **ID1**

$$\nabla\cdot(\phi\overline{A}) = (\nabla\phi)\cdot\overline{A} + \phi\nabla\cdot\overline{A}$$ **ID2**

$$\nabla\times(\phi\overline{A}) = (\nabla\phi)\times\overline{A} + \phi\nabla\times\overline{A}$$ **ID3**

$$\nabla\cdot(\overline{A}\times\overline{B}) = \overline{B}\cdot(\nabla\times\overline{A}) - \overline{A}\cdot(\nabla\times\overline{B})$$ **ID4**

$$\nabla\times(\overline{A}\times\overline{B}) = (\overline{B}\cdot\nabla)\overline{A} - \overline{B}(\nabla\cdot\overline{A}) - (\overline{A}\cdot\nabla)\overline{B} + \overline{A}(\nabla\cdot\overline{B})$$ **ID5**

$$\nabla(\overline{A}\cdot\overline{B}) = (\overline{B}\cdot\nabla)\overline{A} + (\overline{A}\cdot\nabla)\overline{B} + \overline{B}\times(\nabla\times\overline{A}) + \overline{A}\times(\nabla\times\overline{B})$$ **ID6**

$$\nabla\times(\nabla\phi) = 0$$ **ID7**

$$\nabla\cdot(\nabla\times\overline{A}) = 0$$ **ID8**

$$\nabla\times(\nabla\times\overline{A}) = \nabla(\nabla\cdot\overline{A}) - \nabla^2\overline{A}$$ **ID9**

See next slides for the proof of some of these identities

NABLARÄKNING

Let’s consider **ID2**: $$\nabla\cdot\left(\phi\bar{A}\right)=\underbrace{\left(\frac{\partial}{\partial x},\frac{\partial}{\partial z},\frac{\partial}{\partial z}\right)}\cdot\left(\phi\bar{A}\right)$$

This seems almost like a vector!

Can we simply use the vector algebra rules? ***<u>NO!</u>***

Nabla contains derivatives and we know that: $$\frac{d(fg)}{dx}=\frac{df}{dx}g+f\frac{dg}{dx}$$ **ID1**

The derivative must be applied to all the fields in the bracket.
How to remember that with the nabla?

By adding dots to each field and rewriting the expression as a sum:

$$\nabla\cdot\left(\phi\bar{A}\right)=\nabla\cdot\left(\underset{\cdot}{\phi}\bar{A}\right)+\nabla\cdot\left(\phi\underset{\cdot}{\bar{A}}\right)$$

IMPORTANT: after the previous step, the nabla will be applied only to the field with the dot. Now the expression can be rewritten using vector algebra rules (the goal is to obtain an expression in which only the field with the dot follows nabla):

$$\nabla\cdot\left(\phi\bar{A}\right)=\underbrace{\nabla\cdot\left(\underset{\cdot}{\phi}\bar{A}\right)+\nabla\cdot\left(\phi\underset{\cdot}{\bar{A}}\right)}=\underbrace{\bar{A}\cdot\nabla\phi+\phi\nabla\cdot\bar{A}}$$ **ID2**

rewriting the expression using vector algebra

$$\overbrace{\bar{n}\cdot\left(\underset{\cdot}{c}\bar{a}\right)+\bar{n}\cdot\left(c\underset{\cdot}{\bar{a}}\right)}=$$

$$\bar{n}\cdot\left(\bar{a}\right)\underset{\cdot}{c}+\bar{n}\cdot\left(\underset{\cdot}{\bar{a}}\right)c=$$

$$\bar{a}\cdot\bar{n}\,\underset{\cdot}{c}+\bar{n}\cdot\underset{\cdot}{\bar{a}}\,c=\overbrace{\bar{a}\cdot\bar{n}\,\underset{\cdot}{c}+c\,\bar{n}\cdot\underset{\cdot}{\bar{a}}}$$

EXERCISE: prove that $\nabla\times\left(\phi\bar{A}\right)=\left(\nabla\phi\right)\times\bar{A}+\phi\nabla\times\bar{A}$ **ID3**

NABLARÄKNING

To correctly perform the nabla calculation, there are <u>three steps</u>.

We want to calculate the following expression: $\nabla\cdot\cdot\left(\phi, \bar{A}, \psi, \bar{B}, \ldots\right)$

Where $\nabla\cdot\cdot$ *can be:* ∇ *(gradient) or* $\nabla\cdot$ *(divergence) or* $\nabla\times$ *(curl)*

STEP 1 Rewrite the expression as a sum with N terms, where N is the number of (scalar or vector) fields in the expression. Every term in the sum must be identical to the original expression, but the *i-th* field in the *i-th* term must have a dot. This is to remember that nabla is applied to the field with the "dot".

$$\nabla\cdot\cdot\left(\phi, \bar{A}, \psi, \bar{B}, \ldots\right) = \nabla\cdot\cdot\left(\underset{\cdot}{\phi}, \bar{A}, \psi, \bar{B}, \ldots\right) + \nabla\cdot\cdot\left(\phi, \underset{\cdot}{\bar{A}}, \psi, \bar{B}, \ldots\right) +$$
$$\nabla\cdot\cdot\left(\phi, \bar{A}, \underset{\cdot}{\psi}, \bar{B}, \ldots\right) + \nabla\cdot\cdot\left(\phi, \bar{A}, \psi, \underset{\cdot}{\bar{B}}, \ldots\right) + \ldots$$

STEP 2 Now, the nabla can be considered as a vector. Each term can be rewritten using vector algebra rules. The aim is to reach an expression for which in each term only the field with the "dot" appears after the nabla.

STEP 3 Finally, you can remove the "dot".

(But remember that THE NABLA IS NOT A VECTOR)

NABLARÄKNING: EXAMPLES

Prove ID4: $\nabla\cdot\left(\overline{A}\times\overline{B}\right)=\overline{B}\cdot\left(\nabla\times\overline{A}\right)-\overline{A}\cdot\left(\nabla\times\overline{B}\right)$ **ID4**

$$\nabla\cdot\left(\overline{A}\times\overline{B}\right)=\nabla\cdot\left(\underset{\cdot}{\overline{A}}\times\overline{B}\right)+\nabla\cdot\left(\overline{A}\times\underset{\cdot}{\overline{B}}\right)=$$

Now nabla can be treated as vector.

Then, since: $\overline{n}\cdot\left(\overline{A}\times\overline{B}\right)=\overline{B}\cdot\left(\overline{n}\times\overline{A}\right)=-\overline{A}\cdot\left(\overline{n}\times\overline{B}\right)$

$$=\overline{B}\cdot\left(\nabla\times\underset{\cdot}{\overline{A}}\right)-\overline{A}\cdot\left(\nabla\times\underset{\cdot}{\overline{B}}\right)=\overline{B}\cdot rot\,\overline{A}-\overline{A}\cdot rot\,\overline{B}$$

Prove ID7: $\nabla\times\left(\nabla\phi\right)=0$ **ID7**

$$\nabla\times\left(\nabla\phi\right)=\nabla\times\left(\nabla\underset{\cdot}{\phi}\right)=$$

then, since: $\overline{n}\times\left(\overline{n}\lambda\right)=\lambda\left(\overline{n}\times\overline{n}\right)=0$

$$=\nabla\times\left(\nabla\underset{\cdot}{\phi}\right)=0$$

Prove ID9: $\nabla\times\left(\nabla\times\overline{A}\right)=\nabla\left(\nabla\cdot\overline{A}\right)-\nabla^2\overline{A}$ **ID9**

$$\nabla\times\left(\nabla\times\overline{A}\right)=\nabla\times\left(\nabla\times\underset{\cdot}{\overline{A}}\right)=$$

since: $\overline{n}\times\left(\overline{n}\times\overline{c}\right)=\overline{n}(\overline{n}\cdot\overline{c})-\overline{c}(\overline{n}\cdot\overline{n})$

$$=\nabla\left(\nabla\cdot\underset{\cdot}{\overline{A}}\right)-\left(\nabla\cdot\nabla\right)\underset{\cdot}{\overline{A}}=\nabla\left(\nabla\cdot\overline{A}\right)-\nabla^2\overline{A}$$

THE VECTOR LAPLACIAN: general definition

- The scalar Laplacian has been defined as: $\nabla^2\phi = \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right)$

- In a Cartesian coordinate system, the vector Laplacian is defined as:

$$\nabla^2\overline{A} = \left(\nabla^2 A_x\right)\hat{e}_x + \left(\nabla^2 A_y\right)\hat{e}_y + \left(\nabla^2 A_z\right)\hat{e}_z$$

- In any other coordinate system, the vector Laplacian is defined using ID9:

$$\nabla^2\overline{A} = \nabla\left(\nabla\cdot\overline{A}\right) - \nabla\times\left(\nabla\times\overline{A}\right)$$

EXERCISE: calculate $\nabla^2\hat{e}_r$

ELECTROMAGNETIC WAVE EQUATION IN VACUUM

We start from the **Maxwell's equations** in vacuum and in a charge-free space:

$$\nabla \cdot \overline{E} = 0$$

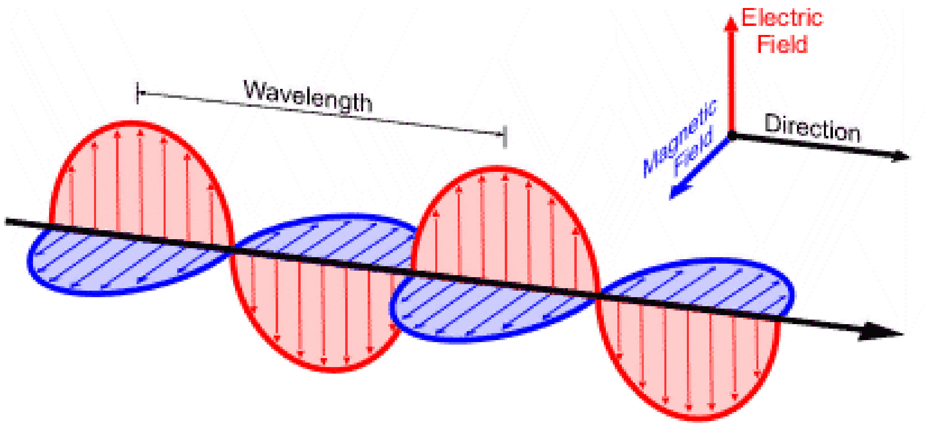
$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t}$$

A magnetic field that varies in time produces an electric field.

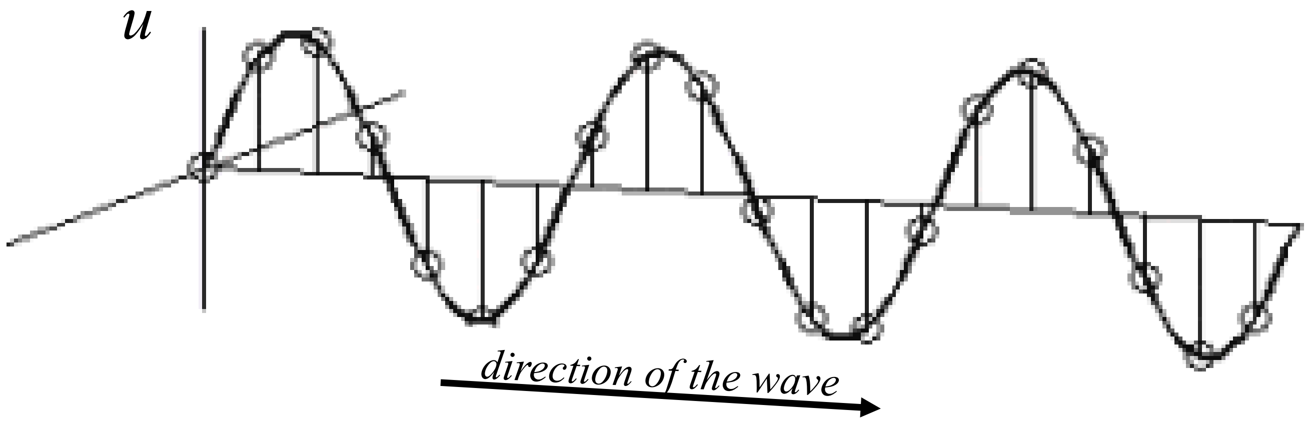
$$\nabla \cdot \overline{B} = 0$$

$$\nabla \times \overline{B} = \mu_0 \varepsilon_0 \frac{\partial \overline{E}}{\partial t}$$

An electric field that varies in time produces a magetic field.



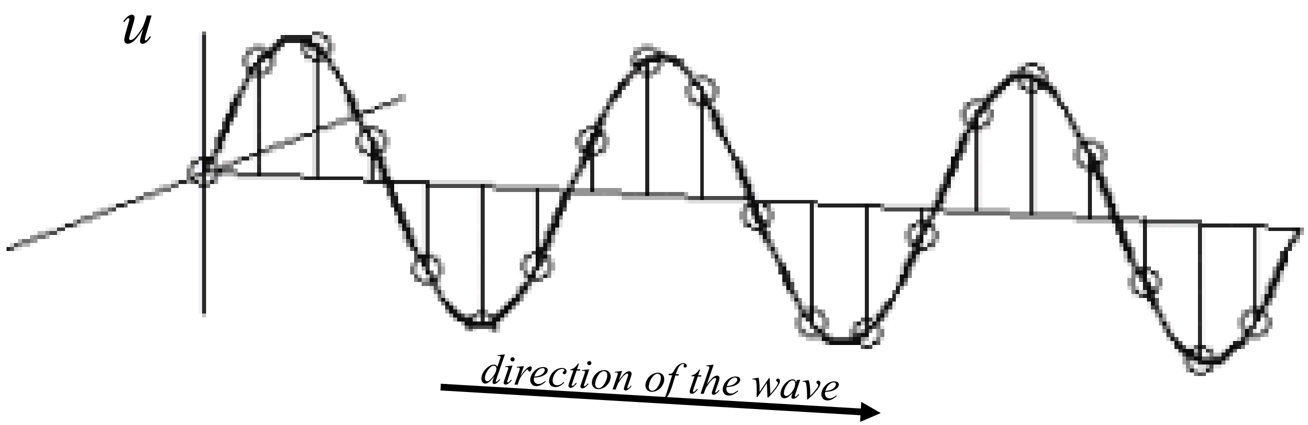
mechanical wave



$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

v is the velocity of the wave

mechanical wave



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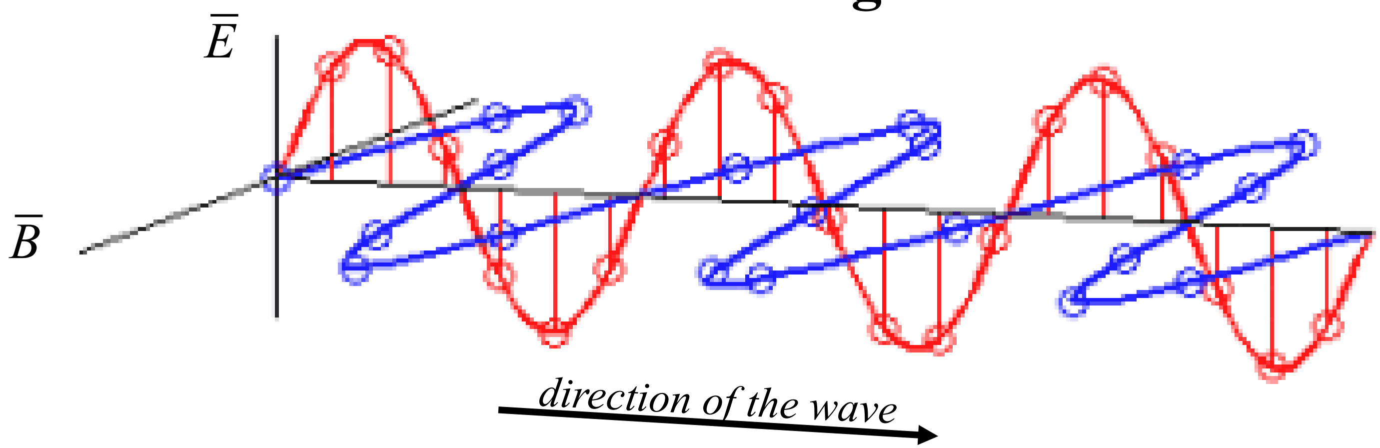

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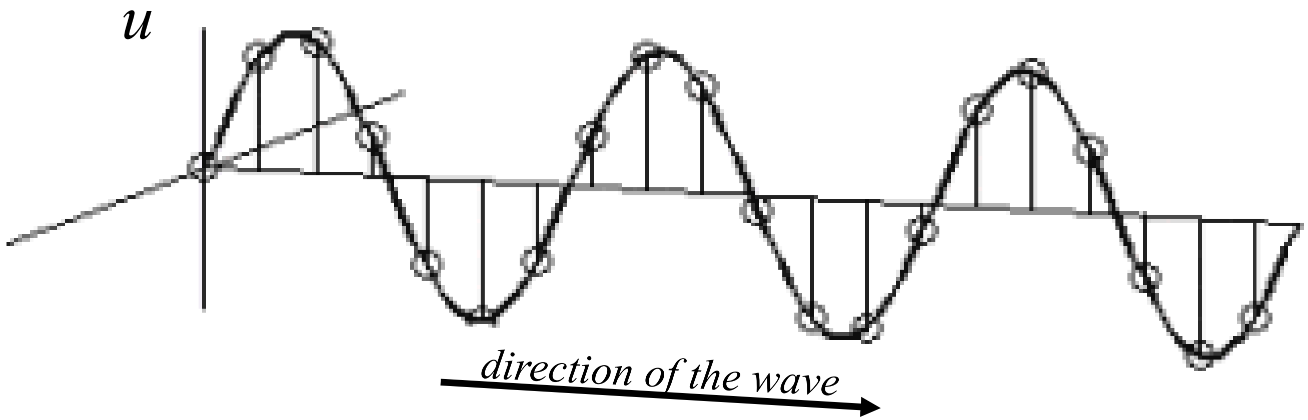

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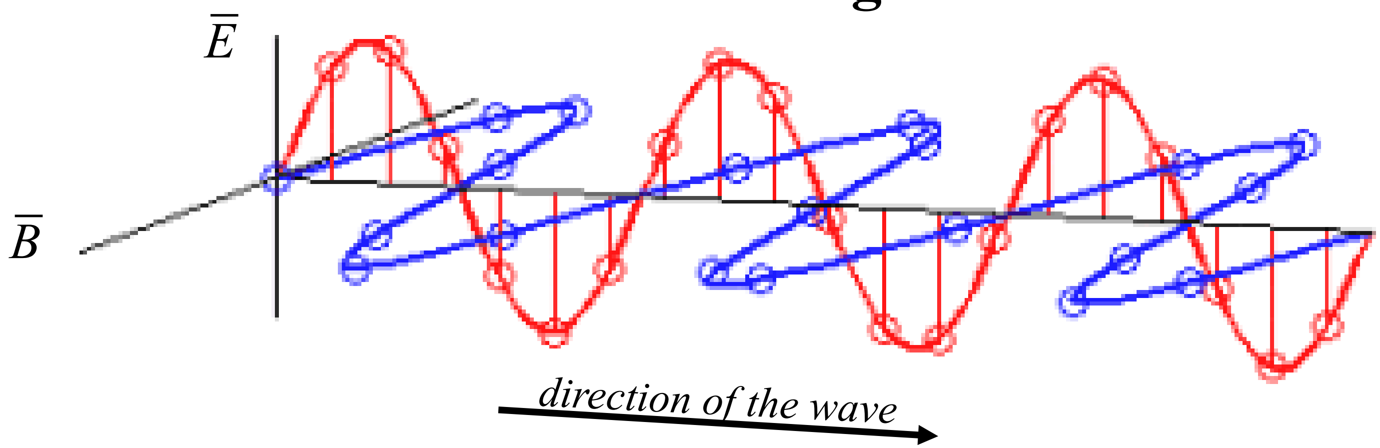

$$\nabla^2 \overline{E} = \frac{1}{v^2}\frac{\partial^2 \overline{E}}{\partial t^2} \quad and \quad \nabla^2 \overline{B} = \frac{1}{v^2}\frac{\partial^2 \overline{B}}{\partial t^2}$$

ELECTROMAGNETIC WAVE EQUATION IN VACUUM

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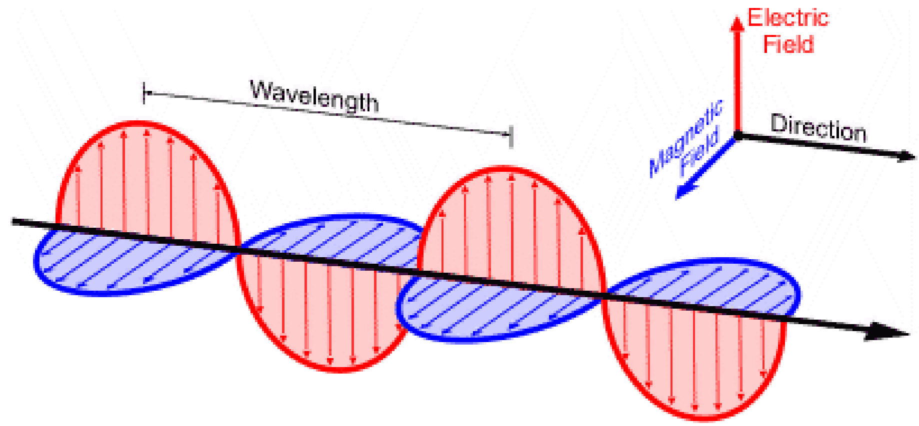

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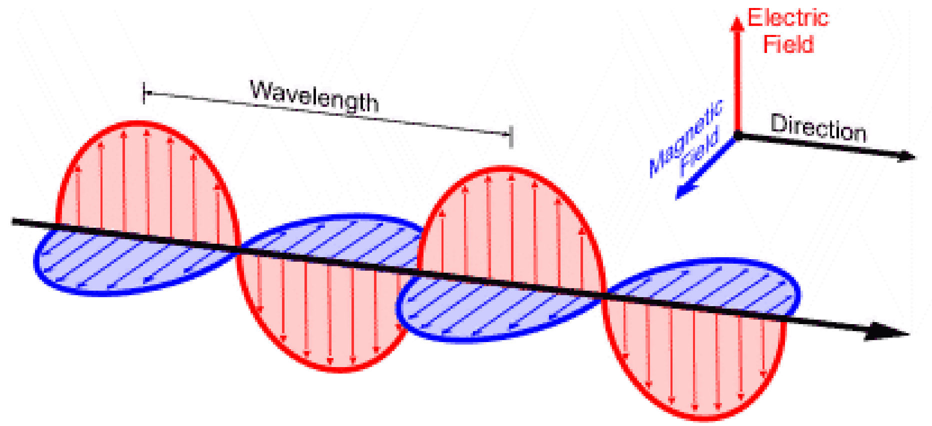


$$\left.\begin{aligned}
&\nabla \times \left(\nabla \times \overline{E}\right) = \nabla\left(\nabla \cdot \overline{E}\right) - \nabla^2 \overline{E} = -\nabla^2 \overline{E} \\
&\nabla \times \left(\nabla \times \overline{E}\right) = \nabla \times \left(-\frac{\partial \overline{B}}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\nabla \times \overline{B}\right) = -\frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial \overline{E}}{\partial t}\right) = -\mu_0 \varepsilon_0 \frac{\partial^2 \overline{E}}{\partial t^2}
\end{aligned}\right\} \Rightarrow \boxed{\nabla^2 \overline{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \overline{E}}{\partial t^2}}$$

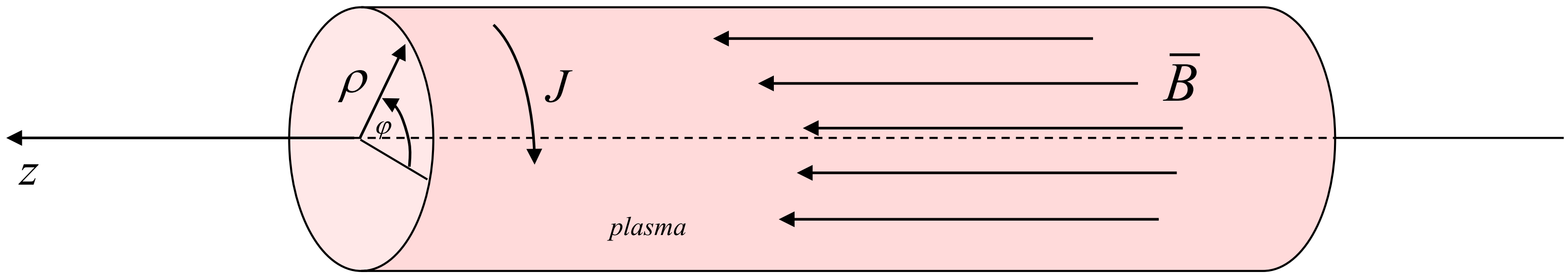
in a similar way, we can obtain: $\boxed{\nabla^2 \overline{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \overline{B}}{\partial t^2}}$

The wave propagates with velocity : $\boxed{v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}}$

$$\left.\begin{aligned}
&\varepsilon_0 = 8.85 \cdot 10^{-12}\, F/m \text{ (vacuum permittivity)} \\
&\mu_0 = 4\pi \cdot 10^{-7}\, N/A^2 \text{ (vacuum permeability)}
\end{aligned}\right\} \Rightarrow v = 2.99 \cdot 10^8\, m/s$$

(which is the velocity of the light in vacuum)

TARGET PROBLEM



$$grad\ \ p = \frac{1}{\mu_0}\left(rot\bar{B}\right)\times\bar{B}$$

$$\nabla p = \frac{1}{\mu_0}\left(\nabla\times\bar{B}\right)\times\bar{B}$$

To go further, we know that: $\bar{a}\times\left(\bar{n}\times\bar{b}\right)=\bar{n}(\bar{a}\cdot\bar{b})-\bar{b}(\bar{a}\cdot\bar{n})$

So, we can express $(\nabla\times\bar{B})\times\bar{B}$ as :

$$\left(\nabla\times\underset{\cdot}{\bar{B}}\right)\times\bar{B} = -\bar{B}\times\left(\nabla\times\underset{\cdot}{\bar{B}}\right) = -\nabla(\bar{B}\cdot\underset{\cdot}{\bar{B}})+\left(\bar{B}\cdot\nabla\right)\underset{\cdot}{\bar{B}} =$$

$$= -\frac{1}{2}\nabla|\bar{B}|^2 + \left(\bar{B}\cdot\nabla\right)\bar{B}$$

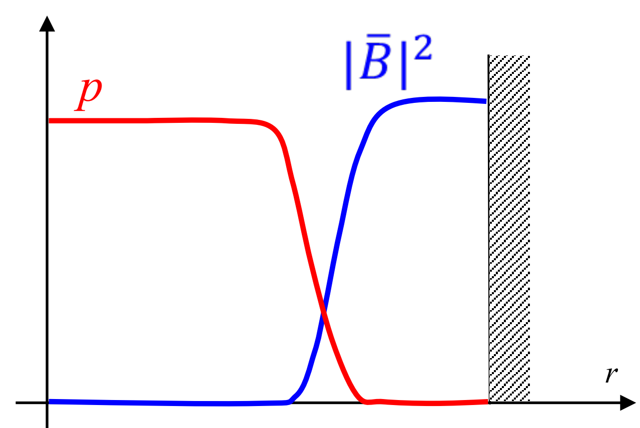
$$\nabla|\bar{B}|^2 = \nabla\left(\bar{B}\cdot\bar{B}\right) = \nabla\left(\underset{\cdot}{\bar{B}}\cdot\bar{B}\right)+\nabla\left(\bar{B}\cdot\underset{\cdot}{\bar{B}}\right) = 2\nabla(\bar{B}\cdot\underset{\cdot}{\bar{B}})$$

$$\boxed{\nabla\left(p+\frac{|\bar{B}|^2}{2\mu_0}\right) = \frac{1}{\mu_0}\left(\bar{B}\cdot\nabla\right)\bar{B}}$$

plasma pressure *magnetic pressure* *Forces due to bending and parallel compression of the field*

In our case field lines are straight and parallel ($\bar{B} = B(\rho)\hat{e}_z$)

$$\nabla\left(p+\frac{|\bar{B}|^2}{2\mu_0}\right) = 0 \qquad \Rightarrow \qquad p+\frac{|\bar{B}|^2}{2\mu_0} = constant$$



A BIT OF HISTORY…

Why the word “nabla”?

The theory of nabla operator was developed by Tait (a co-worker of Maxwell).
It was one of his most important achievements.
Tait was also a good musician in playing an old assyrian instrument similar to an harp.
The name of this instrument in greek is nabla.

The name “nabla operator” was suggested
by James Clerk Maxwell to make a joke on Tait’s hobby

WHICH STATEMENT IS WRONG?

1- grad, div and rot can be expressed using nabla

2- In a curvilinear coordinate system with basis $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$, the vector laplacian can be written as:

$$\nabla^2 \overline{A} = \left(\nabla^2 A_1\right)\hat{e}_1 + \left(\nabla^2 A_2\right)\hat{e}_2 + \left(\nabla^2 A_3\right)\hat{e}_3$$

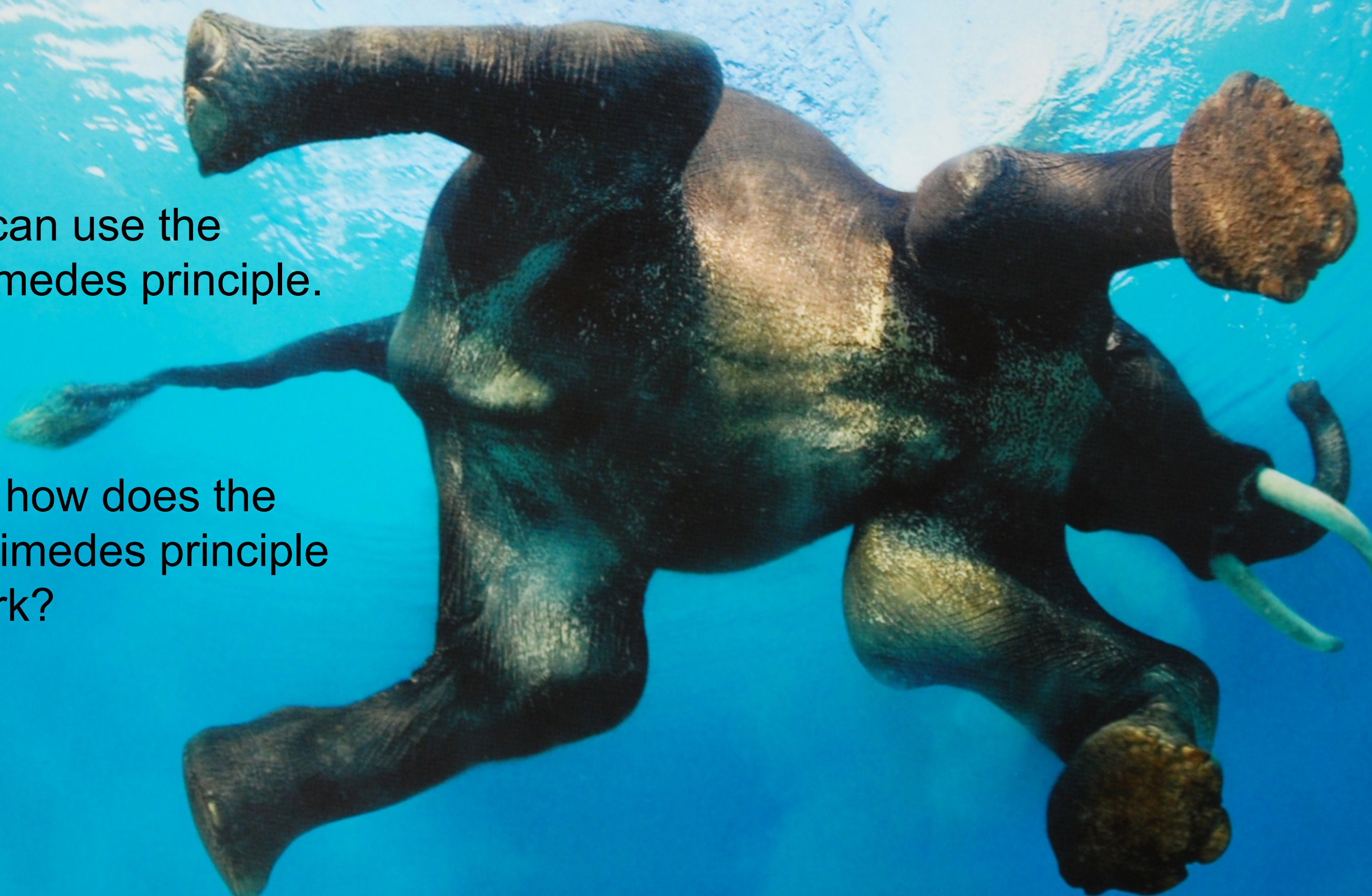
3- $\nabla \times \left(\nabla \phi\right) = 0$

4- $\nabla \cdot \left(\nabla \times \overline{A}\right) = 0$

INTEGRALSATSER

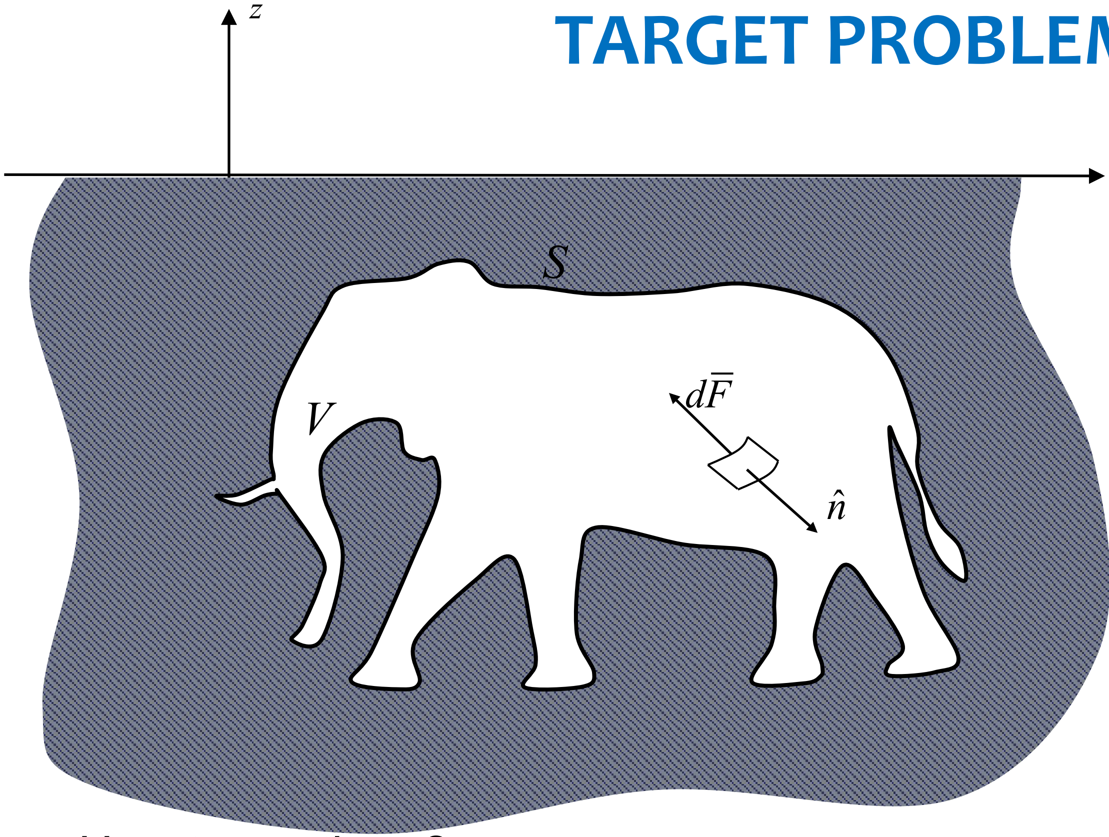
TARGET PROBLEM

- A body is floating in the water
- What is the force that makes it floating?



- We can use the Arkimedes principle.
- But how does the Arkimedes principle work?

TARGET PROBLEM



$$d\bar{F} = -p\hat{n}dS$$

where p [N/m²] is the pressure

$$\bar{F} = \int d\bar{F} = \oiint_S (-p\hat{n}dS) = -\oiint_S pd\bar{S}$$

How to continue?

Apply Gauss's theorem? $\oiint_S \bar{A}\cdot d\bar{S} = \iiint_V div\bar{A}dV$

But $\bar{A}$ is vector,
while p is a scalar!

We need to generalize the Gauss's theorem.

In previous lessons we saw that:

$$\int_P^Q \nabla\phi \cdot d\overline{r} = \phi(Q) - \phi(P) \qquad (1)$$

$$\iint_S \nabla \times \overline{A} \cdot d\overline{S} = \oint_L \overline{A} \cdot d\overline{r} \qquad \textit{(Stokes)} \qquad (2)$$

$$\iiint_V \nabla \cdot \overline{A} dV = \oiint_S \overline{A} \cdot d\overline{S} \qquad \textit{(Gauss)} \qquad (3)$$

What do they have in common?
They all express the integral of a derivative of a function in terms of the values of the function at the integration domain boundaries.

In this sense, theorems (1), (2) and (3) are a generalization of:

$$\int_a^b \frac{df}{dx} dx = f(b) - f(a)$$

We can further generalize the Gauss's theorem :

$$\boxed{\oiint_S d\overline{S}(\ldots) = \iiint_V dV \nabla(\ldots)}$$

Generalized Gauss's theorem

where (...) can be substituted with everything that gives a well defined meaning to both sides of the expression.

EXERCISE: give three examples for the term (...)

$$\boxed{\oiint_S d\bar{S}(\ldots) = \iiint_V dV\nabla(\ldots)}$$

(A) If $(\ldots) = \cdot\bar{A}$, we obtain the Gauss's theorem *(already proved)*

(B) If $(\ldots) = \phi$, we obtain: $\oiint_S d\bar{S}\phi = \iiint_V dV\nabla\phi$

PROOF

$$\hat{e}_x \cdot \iint_S \phi d\bar{S} = \iint_S \phi\hat{e}_x \cdot d\bar{S} \overset{(Gauss)}{=} \iiint_V \nabla(\phi\hat{e}_x)dV \overset{\text{ID2}}{=}$$

$$= \iiint_V \left((\nabla\phi)\cdot\hat{e}_x + \phi\nabla\cdot\hat{e}_x\right)dV = \iiint_V \nabla\phi\cdot\hat{e}_x dV = \hat{e}_x \cdot \iiint_V \nabla\phi dV$$

(C) If $(\ldots) = \times\bar{A}$, we obtain: $\oiint_S d\bar{S}\times\bar{A} = \iiint_V \left(\nabla\times\bar{A}\right)dV$

PROOF

Multiply by $\hat{e}_i$, use the Gauss's theorem and then ID4

We can further generalize also the Stokes' theorem :

$$\boxed{\oint_L d\bar{r}\,(\ldots) = \iint_S \left(d\bar{S} \times \nabla\right)(\ldots)}$$

Generalized Stokes's theorem

where (...) can be substituted with everything that gives a well defined meaning to both sides of the expression.

(A) If $(\ldots) = \cdot\bar{A}$, we obtain the Stokes's theorem *(already proved)*

(B) If $(\ldots) = \phi$, we obtain: $\oint_L \phi d\bar{r} = \iint_S d\bar{S} \times grad\,\phi$

PROOF

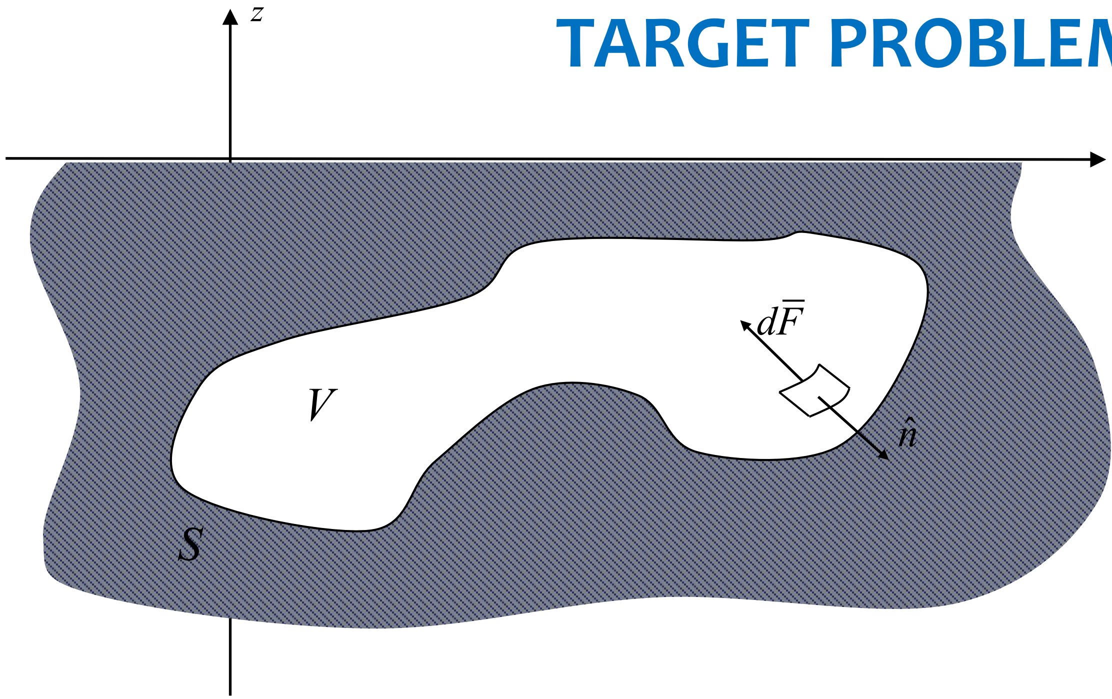
Multiply by $\hat{e}_i$, use the Stokes's theorem and then ID3

(C) If $(\ldots) = \times\bar{A}$, we obtain: $\oint_L d\bar{r} \times \bar{A} = \iint_S \left(d\bar{S} \times \nabla\right) \times \bar{A}$

PROOF

Multiply by $\hat{e}_i$ and use the Stokes's theorem.

TARGET PROBLEM



$$d\bar{F} = -p\hat{n}dS$$

where p [N/m²] is the pressure

$$\bar{F} = \int d\bar{F} = \oint\!\!\oint_S (-p\hat{n}dS) = -\oint\!\!\oint_S pd\bar{S}$$

How to continue?
Apply Gauss's theorem?

$$\oint\!\!\oint_S \bar{A}\cdot d\bar{S} = \iiint_V div\bar{A}dV$$

But $\bar{A}$ is vector,
while p is a scalar!

We apply the generalized Gauss's theorem, with (…)=ϕ.

$$\oint\!\!\oint_S \phi d\bar{S} = \iiint_V \nabla\phi dV$$

$$\left.\begin{aligned} &\bar{F} = -\oint\!\!\oint_S pd\bar{S} = -\iiint_V \nabla p\, dV \\ &p = p_0 - \rho g z \\ &\nabla p = (0, 0, -\rho g) \end{aligned}\right\} \Rightarrow$$

Arkimedes principle

$$\bar{F} = \iiint_V \rho g\hat{e}_z\, dV = \rho g V\hat{e}_z$$

where ρ is the water density and g the gravitational acceleration

WHICH STATEMENT IS WRONG?

1- Gauss and Stokes theorems show that the integral of the derivative of a function is related to the value of the function at the boundary of the integration domain.

2- $\int_L \phi d\overline{r}$ is a vector

3- $\iint_S \phi d\overline{S}$ is a vector

4- $\iint_S d\overline{S} \times \overline{A}$ is a scalar

INDEXRÄKNING

(suffix notation)

AND

(some very basic information on)

CARTESIAN TENSORS

INDEXRÄKNING

To simplify this expression $\nabla \cdot \left(\overline{A} \times \overline{B} \right)$

we used the "nablaräkning"

$$= \nabla \cdot \left(\underset{\cdot}{\overline{A}} \times \overline{B} \right) + \nabla \cdot \left(\overline{A} \times \underset{\cdot}{\overline{B}} \right) = \overline{B} \cdot rot\,\overline{A} - \overline{A} \cdot rot\,\overline{B}$$

Can we use smarter methods?

YES *(sometimes)* !

These are called "suffix notation methods" ("indexräkning") and come from the study of tensors.

To understand this method, we start with a *(brief)* look at Cartesian tensors

PHYSICAL EXAMPLE

ELECTRICAL CONDUCTIVITY

Ohm's law: $\bar{j} = \sigma \bar{E}$

Current density · *Electrical conductivity* · *Electric field*

$$If\ \bar{E} = E_y \hat{e}_y$$

$$then\ \bar{j} = \sigma E_y \hat{e}_y$$

But for many materials this is not true!!

$$\bar{j} = (j_x, j_y, j_z)$$

Is the Ohm's law wrong? NO!

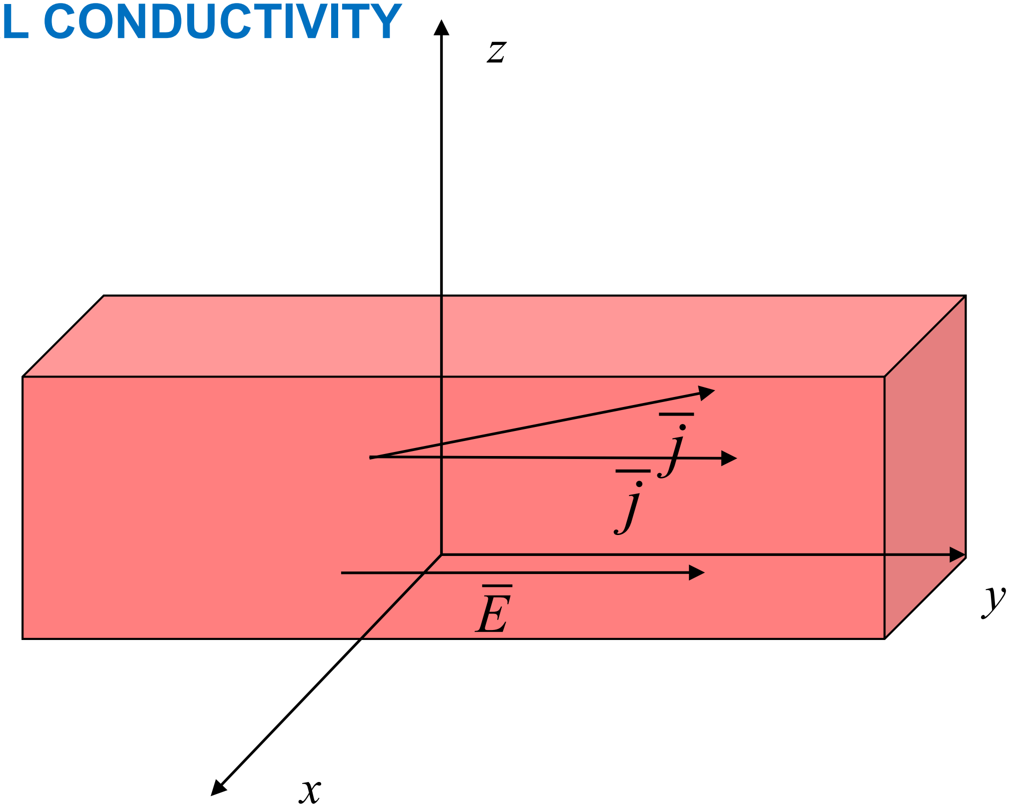
σ is not a scalar

σ is a cartesian tensor of rank 2

$$\bar{j} = \sigma \bar{E} \Rightarrow \begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

$$If\ \bar{E} = (0, E_y, 0)$$

$$then\ \bar{j} = (\sigma_{xy} E_y, \sigma_{yy} E_y, \sigma_{zy} E_y)$$



TENSORS

The Ohm's law is: $\bar{j} = \sigma \bar{E}$

But σ is not a scalar :

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

In suffix notation this can be written very concisely: $j_i = \sigma_{ik} E_k$

σ is a cartesian tensor of rank 2
in the R^3 space.
And it has 3^2 elements

the rank is the number of suffixes

A tensor of rank M
in the R^n space has n^M elements

t_{ij} is a tensor of rank 2 and can be regarded as a matrix
if it is defined in R^2, then $i,j=\{1,2\}$ and it has 2^2 elements
in R^3, then $i,j=\{1,2,3\}$ and it has 3^2 elements
in R^4, then $i,j=\{1,2,3,4\}$ and it has 4^2 elements
…

t_m is a tensor of rank 1 and can be regarded as a vector

A tensor is "Cartesian" if the coordinate system is Cartesian

INDEX NOTATION

1- Indices *x, y, z* can be substituted with *1, 2, 3*

2- Coordinates *x, y, z* with x_1, x_2, x_3.

Examples:

$$A_x = A_1$$

$$\left(A_x, A_y, A_z\right) = \left(A_1, A_2, A_3\right)$$

$$\hat{e}_x = \hat{e}_1$$

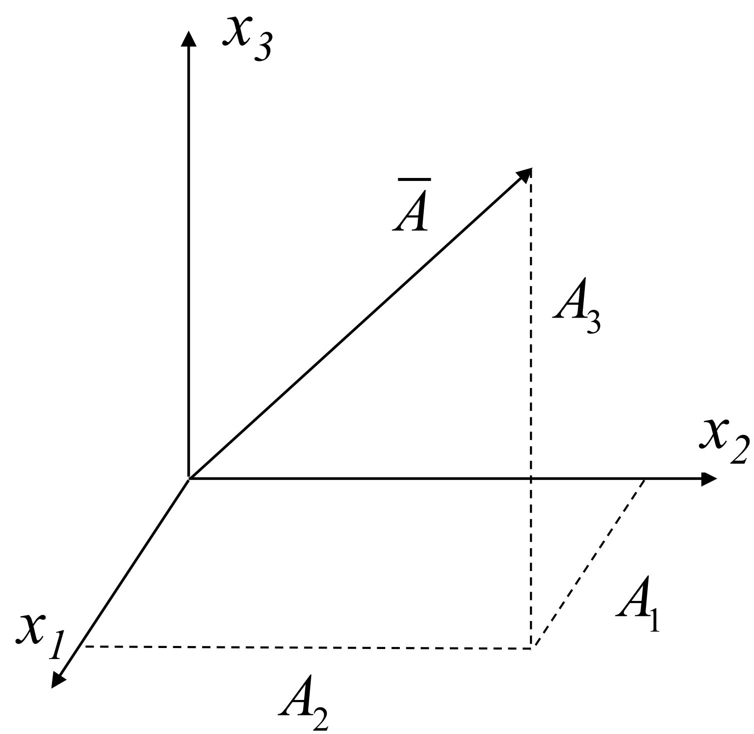
$$\hat{e}_y = \hat{e}_2$$

$$\hat{e}_z = \hat{e}_3$$

$$\frac{\partial \phi}{\partial y} = \partial_2 \phi = \phi_{,2} \qquad \frac{\partial A_x}{\partial y} = A_{1,2}$$

$$\overline{c} = \overline{a} + \overline{b} \quad \Rightarrow \quad \underbrace{c_i = a_i + b_i}$$

in suffix notation this corresponds to the 3 equations obtained using i=1,2,3



The suffix *i* is called "free suffix"

The choice of the free suffix is arbitrary:

$$c_j = a_j + b_j$$

$$c_m = a_m + b_m$$

represent the same equation!

But <u>the same free suffix must be used</u> for each term of the equation

INDEX NOTATION

3- Summation convention:

$$\bar{a}\cdot\bar{b} = a_1b_1 + a_2b_2 + a_3b_3 = \sum_{i=1,3} a_ib_i \qquad \Rightarrow \boxed{\bar{a}\cdot\bar{b} = a_ib_i}$$

whenever a suffix is repeated in a single term in
an equation, summation from 1 to 3 is implied. The repeated suffix is
called *dummy suffix*.

The choice of the dummy suffix is arbitrary: we can write also $\bar{a}\cdot\bar{b} = a_kb_k$

No suffix appears more than twice in any term of the expression:

$$\left(\bar{a}\cdot\bar{b}\right)\left(\bar{c}\cdot\bar{d}\right) = a_ib_i\underbrace{c_jd_j}$$

we cannot use "i" also here!

But the ordering of terms is arbitrary: $a_ib_ic_jd_j = c_ja_id_jb_i = c_ka_md_kb_m = \left(\bar{a}\cdot\bar{b}\right)\left(\bar{c}\cdot\bar{d}\right)$

free suffix

Example: $a_kb_hc_k = a_kc_kb_h = \left(\sum_k a_kc_k\right)b_h = \left[\left(\bar{a}\cdot\bar{c}\right)\bar{b}\right]_h$

dummy suffix

EXERCISE. Write this expression using vectors: $a_ib_ka_nc_ka_i$

The Kronecker delta

The **Kronecker delta** is a tensor of rank 2 defined as:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & otherwise \end{cases}$$

It can be visualized as a nxn identity matrix (where n is the dimension of the space)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Some properties of the Kronecker delta:

$$\boxed{\delta_{ii} = 3}$$

$$\delta_{ii} = \sum_{i=1}^{3} \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$

(in a 3D space)

summation convention

(in space with N dim.)

$$\boxed{\delta_{km} a_m = a_k}$$

$$\delta_{km} a_m = \sum_{m=1}^{N} \delta_{km} a_m = a_1 \delta_{k1} + a_2 \delta_{k2} + \ldots + a_m \delta_{km} + \ldots = a_k$$

$$\boxed{\delta_{km} l_{jm} = l_{jk}}$$

$$l_{jm} \delta_{km} = \sum_{m=1}^{N} l_{jm} \delta_{km} = \underbrace{l_{j1} \delta_{k1} + l_{j2} \delta_{k2} + \ldots + l_{jm} \delta_{km} + \ldots}_{\text{all zeros, unless } k=m} = l_{jk}$$

summation convention

The alternating tensor

(Levi-Civita tensor or permutationssymbolen)

The **alternating tensor** ε_{ijk} *(a tensor of rank 3)* is defined as:

$$\varepsilon_{ijk} = \hat{e}_i \cdot \left(\hat{e}_j \times \hat{e}_k\right) = \begin{cases} 0 & \textit{if any of } i,j,k \textit{ are equal} \\ +1 & \textit{if } (i,j,k) = (1,2,3) \textit{ or } (2,3,1) \textit{ or } (3,1,2) \quad (\textit{even permutation of } 1,2,3) \\ -1 & \textit{if } (i,j,k) = (1,3,2) \textit{ or } (2,1,3) \textit{ or } (3,2,1) \quad (\textit{odd permutation of } 1,2,3) \end{cases}$$

The alternating tensor can be used to express the cross product: $\boxed{\left(\bar{a} \times \bar{b}\right)_i = \varepsilon_{ijk} a_j b_k}$

PROOF:

$$\left(\bar{a} \times \bar{b}\right)_i = \hat{e}_i \cdot \left(\bar{a} \times \bar{b}\right) = \hat{e}_i \cdot \left[\left(a_j \hat{e}_j\right) \times \left(b_k \hat{e}_k\right)\right] = \hat{e}_i \cdot \left(\hat{e}_j \times \hat{e}_k\right) a_j b_k = \varepsilon_{ijk} a_j b_k$$

EXAMPLE FOR THE x COMPONENT *(i=1)*:

$$\left(\bar{a} \times \bar{b}\right)_1 = a_2 b_3 - a_3 b_2$$

$$\varepsilon_{1jk} a_j b_k = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{1jk} a_j b_k = \varepsilon_{123} a_2 b_3 + \varepsilon_{132} a_3 b_2 = a_2 b_3 - a_3 b_2$$

Some properties:

$$\boxed{\begin{aligned} &\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} \\ &\varepsilon_{ijk} = -\varepsilon_{jik} \\ &\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \end{aligned}}$$

(any even permutation of i,j,k do NOT change the sign)

(any odd permutation of i,j,k changes the sign)

Very useful to simplify expressions involving two cross products

GRADIENT, DIVERGENCE AND CURL IN INDEX NOTATION

GRADIENT

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = \left(\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \frac{\partial\phi}{\partial x_3}\right) = \left(\phi_{,1}, \phi_{,2}, \phi_{,3}\right)$$

So, the component i of the gradient is: $\boxed{\left(\nabla\phi\right)_i = \phi_{,i}}$

DIVERGENCE

$$\nabla\cdot\overline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} = \sum_i A_{i,i} = A_{i,i}$$

So, the divergence is: $\boxed{\nabla\cdot\overline{A} = A_{i,i}}$

CURL

$$\left(\nabla\times\overline{A}\right)_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} =$$

$$A_{3,2} - A_{2,3} = \varepsilon_{123}A_{3,2} + \varepsilon_{132}A_{2,3} = \varepsilon_{1jk}A_{k,j}$$

So, the component i of the curl is: $\boxed{\left(\nabla\times\overline{A}\right)_i = \varepsilon_{ijk}A_{k,j}}$

"Nablaräkning" and "Indexräkning"

use of tensors in the calculation of nabla expressions

Calculate: $\nabla\cdot\left(\overline{a}\times\overline{r}\right)$ where $\boxed{\overline{r}=\left(x,y,z\right)}$ and $\overline{a}$ is constant

1- Nablaräkning

$\overline{n}\cdot\left(\overline{a}\times\overline{b}\right)=\overline{a}\cdot\left(\overline{b}\times\overline{n}\right)$

$$\nabla\cdot\left(\overline{a}\times\overline{r}\right)=\nabla\cdot\left(\underset{\cdot}{\overline{a}}\times\overline{r}\right)+\nabla\cdot\left(\overline{a}\times\underset{\cdot}{\overline{r}}\right)=0+\overline{a}\cdot\left(\underset{\cdot}{\overline{r}}\times\nabla\right)=-\overline{a}\cdot\underbrace{\left(\nabla\times\overline{r}\right)}_{=0}=0$$

$\overline{a}$ *is a constant*

2- Indexräkning

$$\nabla\cdot\left(\overline{a}\times\overline{r}\right)=\left(\varepsilon_{ikl}a_k r_l\right)_{,i}=\varepsilon_{ikl}\left(a_{k,i}r_l+a_k r_{l,i}\right)=\varepsilon_{ikl}a_k r_{l,i}=0$$

$r_{l,i}\neq 0$ *only if* $l=i$

If $l=i$ *then* $\varepsilon_{ijk}=0$

INDEXRÄKNING

Prove that:

$$\boxed{\bar{a}\cdot\left(\bar{b}\times\bar{c}\right)=-\bar{b}\cdot\left(\bar{a}\times\bar{c}\right)}$$

$$\bar{a}\cdot\left(\bar{b}\times\bar{c}\right)=a_i\left(\bar{b}\times\bar{c}\right)_i=a_i\varepsilon_{ijk}b_jc_k=b_j\varepsilon_{ijk}a_ic_k=-b_j\varepsilon_{jik}a_ic_k=-b_j\left(\bar{a}\times\bar{c}\right)_j=-\bar{b}\cdot\left(\bar{a}\times\bar{c}\right)$$

Prove that:

$$\boxed{\nabla\cdot\left(\phi\bar{A}\right)=\nabla\phi\cdot\bar{A}+\phi\nabla\cdot\bar{A}}$$

$$\left.\begin{array}{l}\nabla\cdot\underbrace{\left(\phi\bar{A}\right)}_{\bar{v}}\\ \nabla\cdot\bar{v}=v_{i,i}\\ v_i=\left(\phi A\right)_i=\phi A_i\end{array}\right\}\Rightarrow\nabla\cdot\left(\phi\bar{A}\right)=\left(\phi\bar{A}\right)_{i,i}=\left(\phi A_i\right)_{,i}=\underbrace{\phi_{,\mathrm{i}}}_{(\nabla\phi)_{,i}}A_i+\phi A_{i,i}=\nabla\phi\cdot\bar{A}+\phi\nabla\cdot\bar{A}$$

Prove that:

$$\boxed{\nabla\times\left(\phi\bar{A}\right)=\nabla\phi\times\bar{A}+\phi\left(\nabla\times\bar{A}\right)}$$

$$\left.\begin{array}{l}\nabla\times\underbrace{\left(\phi\bar{A}\right)}_{\bar{v}}\\ \left(\nabla\times\bar{v}\right)_i=\varepsilon_{ijk}v_{k,\mathrm{j}}\\ v_k=\left(\phi A\right)_k=\phi A_k\end{array}\right\}\Rightarrow\left(\nabla\times\left(\phi\bar{A}\right)\right)_i=\varepsilon_{ijk}\left(\phi A_k\right)_{,\mathrm{j}}=\varepsilon_{ijk}\underbrace{\phi_{,\mathrm{j}}}_{(\nabla\phi)_j}A_k+\varepsilon_{ijk}\phi A_{k,\mathrm{j}}=\left(\nabla\phi\times\bar{A}\right)_i+\phi\left(\nabla\times\bar{A}\right)_i$$