VEKTORANALYS

HT 2021

CELTE / CENMI

ED1110

CURVILINEAR COORDINATE SYSTEMS (kroklinjiga koordinatsytem)

ÖVNINGAR

Kursvecka 4

Kapitel 10 (*Vektoranalys,* 1:e uppl, Frassinetti/Scheffel)

KTH VETENSKAP OCH KONST

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PROBLEM 1

Calculate in spherical coordinates: $grad\left(div(\hat{e}_r)\right) - rot\left(rot(\hat{e}_r)\right)$

IMPORTANT COMMENT (but not useful to solve the problem)

as we will discuss next week: $grad\left(div\left(\overline{A}\right)\right) - rot\left(rot\left(\overline{A}\right)\right) = \nabla\left(\nabla \cdot \overline{A}\right) - \nabla \times \left(\nabla \times \overline{A}\right) = \nabla^2 \overline{A}$

$\nabla^2 \overline{A}$ is called **"vector laplacian"** (see Section 14.2 in the book)

SOLUTION to problem 1

$$grad\left(div(\hat{e}_r)\right) - rot\left(rot(\hat{e}_r)\right)$$

We need gradient, divergence and curl in spherical coordinates

SOLUTION to problem 1

Spherical coordinates: r, θ, φ

$$\overline{r} = r\sin\theta\cos\varphi\hat{e}_x + r\sin\theta\sin\varphi\hat{e}_y + r\cos\theta\hat{e}_z$$

position vector using spherical coordinates
<u>but</u> in a cartesian coordinate system

$$\begin{cases} h_r = 1 \\ h_\theta = r \\ h_\varphi = r\sin\theta \end{cases}$$

scale factors of a spherical coordinate system

$$grad\,\phi = \left(\frac{1}{h_1}\frac{\partial\phi}{\partial u_1}, \frac{1}{h_2}\frac{\partial\phi}{\partial u_2}, \frac{1}{h_3}\frac{\partial\phi}{\partial u_3}\right) = \left(\frac{1}{h_r}\frac{\partial\phi}{\partial r}, \frac{1}{h_\theta}\frac{\partial\phi}{\partial\theta}, \frac{1}{h_\varphi}\frac{\partial\phi}{\partial\varphi}\right) = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\right)$$

$$\boxed{grad\,\phi = \frac{\partial\phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\hat{e}_\varphi}$$

$$div\overline{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(A_1h_2h_3\right) + \frac{\partial}{\partial u_2}\left(A_2h_3h_1\right) + \frac{\partial}{\partial u_3}\left(A_3h_1h_2\right)\right] =$$

$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(A_r r^2\sin\theta\right) + \frac{\partial}{\partial\theta}\left(A_\theta r\sin\theta\right) + \frac{\partial}{\partial\varphi}\left(A_\varphi r\right)\right]$$

$$\boxed{div\overline{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(A_\varphi\right)}$$

SOLUTION to problem 1

$$rot\bar{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\hat{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ A_r & rA_\theta & r\sin\theta A_\varphi \end{vmatrix}$$

$$\boxed{rot\bar{A} = \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\varphi\right) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\varphi}\right)\hat{e}_r + \left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\varphi\right)\right)\hat{e}_\theta + \left(\frac{1}{r}\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right)\hat{e}_\varphi}$$

Let’s go back to our initial problem:

$$\nabla^2 \hat{e}_r = grad\left(div\,\hat{e}_r\right) - rot\left(rot\,\hat{e}_r\right)$$

$$\hat{e}_r = (1,0,0) \qquad \textit{(in the spherical coordinate system)}$$

So we have to calculate:

$$div\bar{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(A_\varphi\right)$$

$$with \quad \bar{A} = \hat{e}_r = (1,0,0)$$

SOLUTION to problem 1

Therefore: $$div\,\hat{e}_r = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\right) = \frac{2}{r}$$

Then we have to calculate:

$$rot\overline{A} = \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\varphi\right) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\varphi}\right)\hat{\mathrm{e}}_r + \left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\varphi\right)\right)\hat{\mathrm{e}}_\theta + \left(\frac{1}{r}\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right)\hat{\mathrm{e}}_\varphi$$

$$with\ \overline{A} = \hat{e}_r = (1,0,0)$$

Therefore: $$rot\,\hat{e}_r = \left(0,0,0\right) = \overline{0}$$

$$grad\,\phi = \frac{\partial\phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\hat{e}_\varphi$$

Finally:

$$\nabla^2\hat{e}_r = grad\left(div\,\hat{e}_r\right) - rot\left(rot\,\hat{e}_r\right) = grad\left(\frac{2}{r}\right) - rot(\overline{0}) = -\frac{2}{r^2}\hat{e}_r + 0\hat{e}_\theta + 0\hat{e}_\varphi = -\frac{2}{r^2}\hat{e}_r$$

PROBLEM 2

Consider the following fields: $\psi = -\dfrac{\cos\theta}{r^2}$, $\quad \bar{A} = \dfrac{\sin\theta}{r^2}\hat{e}_\varphi$

where r, θ, φ are spherical coordinates.
Calculate:

$a)\quad grad(\psi)$

$b)\quad rot(\bar{A})$

$c)\quad div(grad(\psi)) \quad and \quad rot(rot(\bar{A}))$

<u>IMPORTANT COMMENT</u> (but not useful to solve the problem)

as we will discuss next week:

$div(grad(\psi)) \equiv \nabla^2\psi$

$\nabla^2\psi$ is called **" Laplacian"** (see Section 14.1 in the book)

SOLUTION to problem 2

(a) $$grad\psi = \frac{\partial\psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}\hat{e}_\varphi = \frac{2\cos\theta}{r^3}\hat{e}_r + \frac{\sin\theta}{r^3}\hat{e}_\theta$$

(b) $$rot(\bar{A}) = \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\varphi) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\varphi}\right)\hat{e}_r + \left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi)\right)\hat{e}_\theta + \left(\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right)\hat{e}_\varphi$$

$$with\ \bar{A} = \frac{\sin\theta}{r^2}\hat{e}_\varphi$$

SOLUTION to problem 2

$$rot\bar{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin^2\theta}{r^2}\right)\hat{e}_r - \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\sin\theta}{r}\right)\hat{e}_\theta + 0\,\hat{e}_\varphi = \frac{1}{r^3\sin\theta}2\sin\theta\cos\theta\,\hat{e}_r + \frac{\sin\theta}{r}\frac{1}{r^2}\hat{e}_\theta =$$

$$\frac{2\cos\theta}{r^3}\hat{e}_r + \frac{\sin\theta}{r^3}\hat{e}_\theta = grad\psi$$

(c) $$\nabla^2\psi = div\left(grad\left(\psi\right)\right) = div\left(rot\left(\bar{A}\right)\right) = 0$$

You need to calculate the divergence of the curl.
The calculation is long, but straightforward.

$$rot\left(rot\left(\bar{A}\right)\right) = rot\left(grad\left(\psi\right)\right) = 0$$

You need to calculate the curl of the gradient.
The calculation is long, but straightforward.

This is a general result:
- *the divergence of the curl is zero.*
- *the curl of the gradient is zero.*

THE RING NEBULA

is a planetary nebula in the northern constellation of Lyra

PROBLEM 3

The Ring nebula can be considered as a sphere with constant radius R_0.
Assume that the velocity of the Helium atoms in the nebula is described by the following field:

$$\overline{v} = k\left(r - \frac{r^2}{2}\cos\varphi\right)\hat{e}_r + k\left(2r^2 \sin\theta \sin\varphi\right)\hat{e}_\varphi$$

where k is a constant.
Calculate the flux of Helium atoms that flows outside the nebula

SOLUTION

$$Flux = \iint_S \overline{v} \cdot d\overline{S} \overset{\text{Guass theorem}}{=} \iiint_V div(\overline{v})dV$$

$$div(\overline{v}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(v_\varphi\right) \quad \textit{(in a spherical coord. system)}$$

$$div\left(k\left(r - \frac{r^2}{2}\cos\varphi\right)\hat{e}_r + 2r^2 k \sin\theta \sin\varphi \hat{e}_\varphi\right) = k\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3 - \frac{r^4}{2}\cos\varphi\right) + k\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(2r^2\sin\theta\sin\varphi\right) =$$

$$= \frac{k}{r^2}\left(3r^2 - 4\frac{r^3}{2}\cos\varphi\right) + k_2\frac{2r^2\sin\theta}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(\sin\varphi\right) = 3k - 2kr\cos\varphi + 2kr\cos\varphi = 3k$$

$$\iint_S \overline{v} \cdot d\overline{S} = \iiint_V 3k dV = 3\frac{4\pi}{3}R_0^3 = 4\pi k R_0^3$$

PROBLEM 4

Kolven i en bilmotor har formen av en cirkulär cylinder med radie $\rho_0=1$ och höjd $z_0=2$. När motorn är igång, fördelar sig temperaturen i cylindern enligt:

$$T = 1 + \rho^2 \left(1 + \cos\varphi\right)\left(z+1\right)$$

(a) Beräkna temperaturgradienten. Beräkna den största temperaturvariationen som funktion av ρ för $\phi=\pi/2$ och $z=0$.

(b) Betrakta punkten P: $(1, \pi/2, 0)$ i cylinderkoordinater. Beräkna riktningen $\bar{v}_M$ (i cylinderkoordinater) i vilken temperaturen ökar snabbast

(c) Beräkna i punkten P en riktning $\hat{v}_1$ i planet $z=0$ längs vilken temperaturändringen är 1.

(d) Beräkna vinkeln mellan $\bar{v}_M$ och $\hat{v}_1$

SOLUTION to problem 4

(a) $T = 1 + \rho^2 (1 + \cos\varphi)(z+1)$

$$grad(T) = \frac{\partial T}{\partial \rho}\hat{e}_\rho + \frac{1}{\rho}\frac{\partial T}{\partial \varphi}\hat{e}_\varphi + \frac{\partial T}{\partial z}\hat{e}_z = 2\rho(1+\cos\varphi)(z+1)\hat{e}_\rho - \rho\sin\varphi(z+1)\hat{e}_\varphi + \rho^2(1+\cos\varphi)\hat{e}_z$$

$at \quad \varphi = \pi/2, z = 0:$

$$grad(T)\big|_{\varphi=\pi/2, z=0} = 2\rho\,\hat{e}_\rho - \rho\,\hat{e}_\varphi + \rho^2\,\hat{e}_z$$

$\Rightarrow$ maximum increase: $|grad(T)| = \sqrt{(2\rho)^2 + (-\rho)^2 + (\rho^2)^2} = \sqrt{5\rho^2 + \rho^4}$

the maximum is at ρ=1

(b) $v_M = grad(T)\big|_P = 2\,\hat{e}_\rho - \hat{e}_\varphi + \hat{e}_z$

(c) $\hat{v}_1$ is on the plane z=0 $\Rightarrow$ $\hat{v}_1 = a\hat{e}_\rho + b\hat{e}_\varphi$ with $\sqrt{a^2+b^2} = 1$

Directional derivative (see section 4.3)

This is the gradient, not the gradient normalized to 1

$$\frac{dT}{ds} = grad(T)\Big|_P \cdot \hat{v}_1 = 1 \;\Rightarrow\; (2\,\hat{e}_\rho - \hat{e}_\varphi + \hat{e}_z)\cdot(a\hat{e}_\rho + b\hat{e}_\varphi) = (2,-1,1)\cdot(a,b,0) = 1 \;\Rightarrow\; \begin{cases} 2a - b = 1 \\ \sqrt{a^2+b^2} = 1 \end{cases} \;\Rightarrow$$

$$\Rightarrow \quad b = 2a - 1 \Rightarrow \sqrt{a^2 + (2a-1)^2} = 1 \Rightarrow 5a^2 - 4a = 0$$

possible solution is a=0 $\Rightarrow b = -1$

$$\Rightarrow \hat{v}_1 = (0,-1,0) = -\hat{e}_\varphi$$

(d)
$$\left.\begin{aligned} v_M \cdot \hat{v}_1 &= (2\,\hat{e}_\rho - \hat{e}_\varphi + \hat{e}_z)\cdot(-\hat{e}_\varphi) = (2,-1,1)\cdot(0,-1,0) = 1 \\ v_M \cdot \hat{v}_1 &= |v_1||\hat{v}_0|\cos\alpha = \sqrt{6}\cos\alpha \end{aligned}\right\} \Rightarrow \alpha = \arccos\frac{1}{\sqrt{6}}$$