VEKTORANALYS

HT 2021

CELTE / CENMI

ED1110

CURVILINEAR COORDINATE SYSTEMS (kroklinjiga koordinatsytem)

Kursvecka 4

Kapitel 10 (*Vektoranalys*, 1:e uppl, Frassinetti/Scheffel)

version: 18-sept-2021

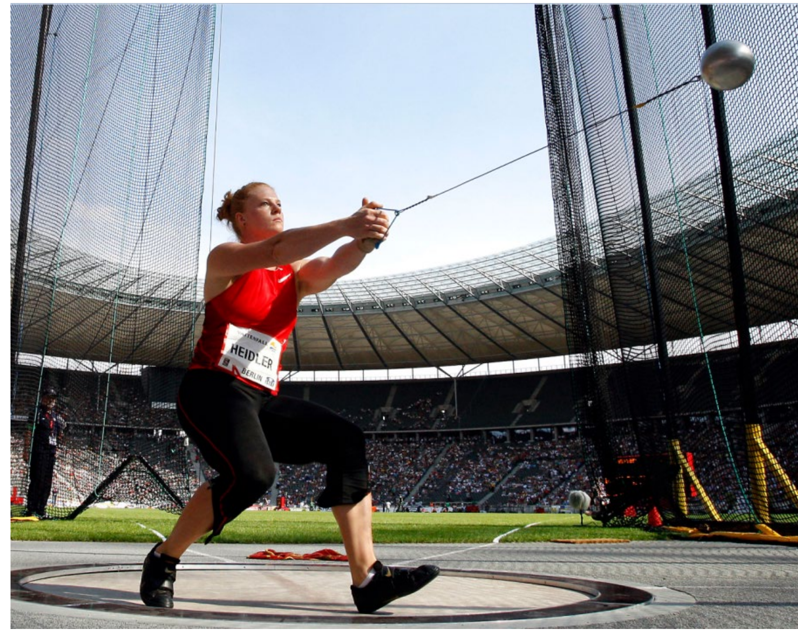
This week

- **Curvilinear coordinate systems:**
  - Basics of any curvilinear coordinate systems:
    - basis vectors
    - differential elements
    - derivative of the basis vectors
  - Specific examples:
    - cylindrical coordinate system
    - sperical coordinate system
- **Gradient, divergence and curl in a curvilinear coordinate system**
  - grad, div, rot in a generig curvilinear coordinate systems:
  - Specific examples:
    - grad, div, rot in a cylindrical coordinate system
    - grad div, rot in a sperical coordinate system

Connections with previous and next topics

- It is a generalization of cylindrical and spherical coordinate systems (week 1)
- It will allow to calculate gradient, divergence and curl in coordinate systems that are not cartesian (week 1 and 3)
- It will introduce part of the formalism necessary to solve Laplace's and Poission's equations (week 6)

TARGET PROBLEM



- An athlete is rotating a hammer
- Calculate the force on the arms of the athlete.

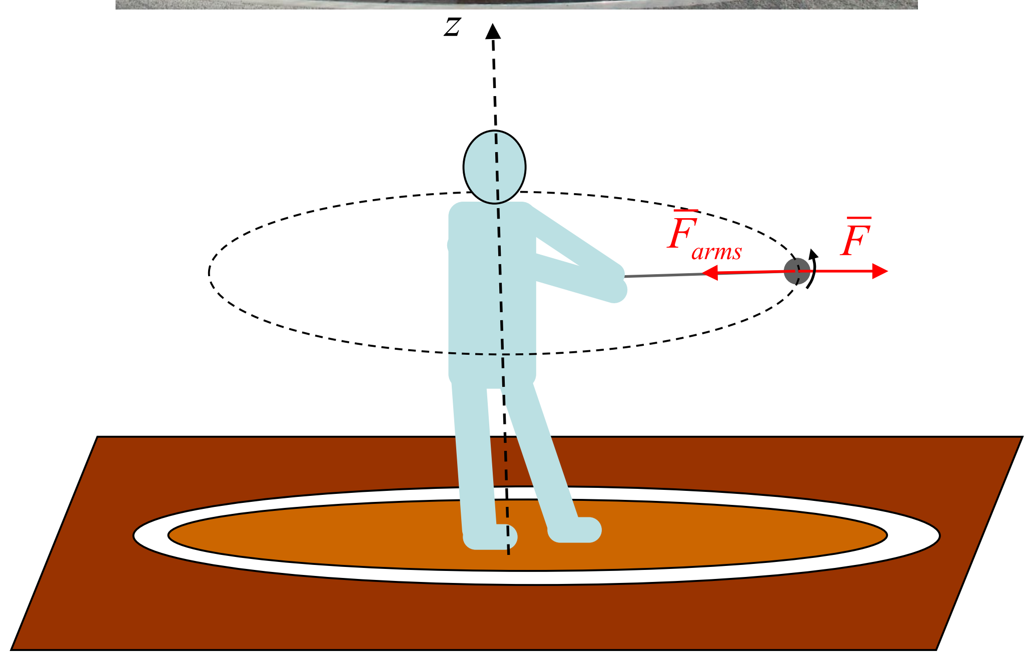
$$\overline{F}_{arms} = -\overline{F}$$

$$\overline{F} = m\overline{a}$$

$$\overline{a} = \frac{d\overline{v}}{dt} \equiv \dot{\overline{v}}$$

$$\overline{v} = \frac{d\overline{r}}{dt} \equiv \dot{\overline{r}}$$

$\overline{r}$ *(in cylindrical coord.)*



We need

- to introduce **curvilinear coordinates**
- to describe **cylindrical coordinates**
- to calculate **the derivative of** $\hat{e}_\rho$

CYLINDRICAL COORDINATES

Cylindrical coordinates are an example of curvilinear coordinates

cartesian coord.

$P:\ x,\ y,\ z$

cylindrical coord.

$P:\ \rho,\ \varphi,\ z$

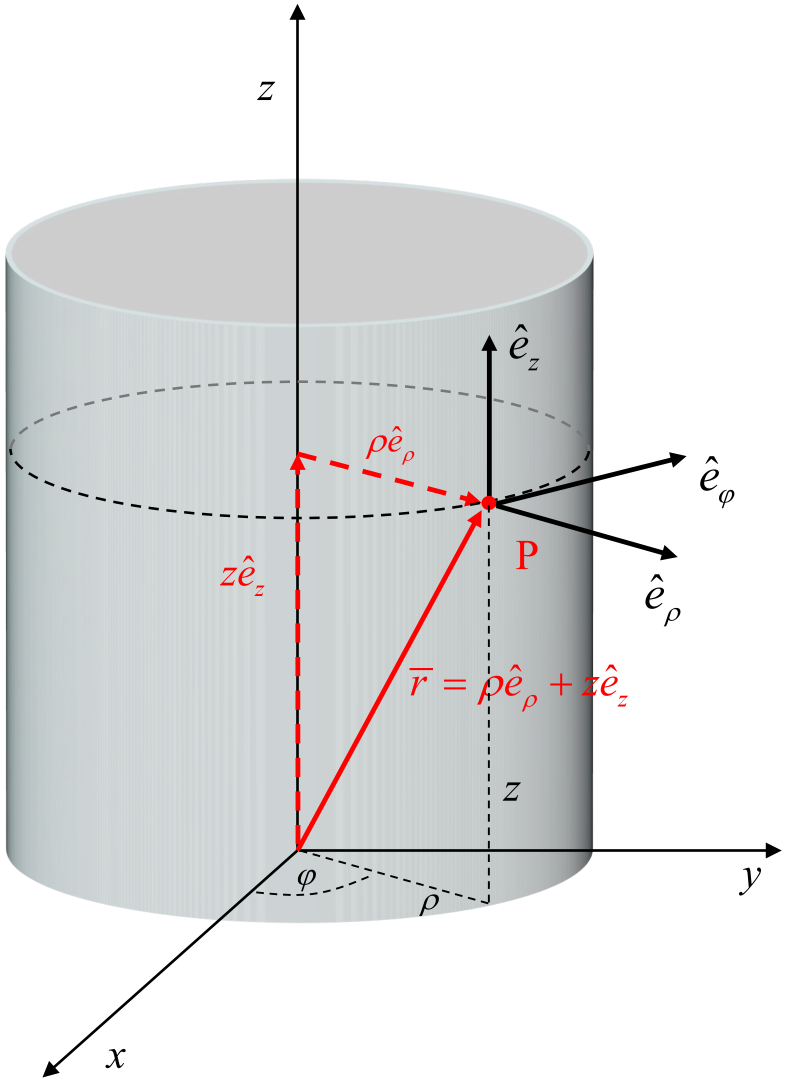
$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases}$$

$$0 \le \rho \le \infty$$

$$0 \le \varphi \le 2\pi$$

$$-\infty \le z \le +\infty$$

$$\begin{cases} \rho = \sqrt{x^2 + y^2} \\ \tan\varphi = y / x \\ z = z \end{cases}$$



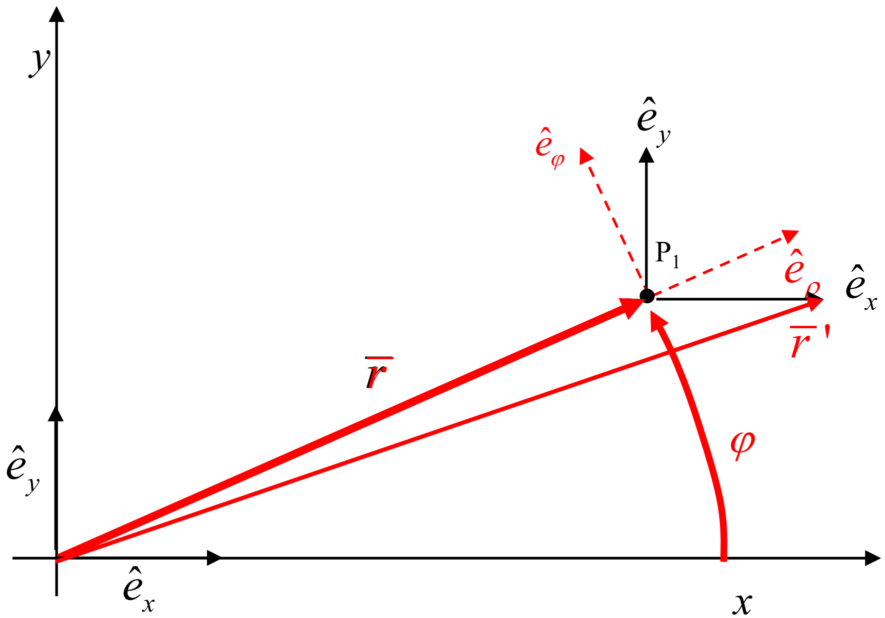
CYLINDRICAL COORDINATE SYSTEM: “informal introduction”

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases}$$

$$\overline{r} = (x, y, z) = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$$

$$\overline{r} = (\rho\cos\varphi, \rho\sin\varphi, z) =$$
$$= \rho\cos\varphi\,\hat{e}_x + \rho\sin\varphi\,\hat{e}_y + z\,\hat{e}_z$$

(in a cartesian coordinate system)



$$\frac{\partial\overline{r}}{\partial x} = \frac{\partial(x, y, z)}{\partial x} = (1, 0, 0) = \hat{e}_x$$

This is the rate of variation of $\overline{r}$ with x keeping y and z constant⇒ it is a vector parallel to the x-axis

$$\frac{\partial\overline{r}}{\partial y} = \frac{\partial(x, y, z)}{\partial y} = (0, 1, 0) = \hat{e}_y$$

This is the rate of variation of $\overline{r}$ with y keeping x and z constant⇒ it is a vector parallel to the x-axis

How to express the $\hat{e}_\rho$ and $\hat{e}_\varphi$ axis of a cylindrical coordinate system in a cartesian coordinate system?

$$\overline{e}_\rho = \frac{\partial\overline{r}}{\partial\rho} = \frac{\partial(\rho\cos\varphi\,\hat{e}_x + \rho\sin\varphi\,\hat{e}_y + z\,\hat{e}_z)}{\partial\rho} = \cos\varphi\hat{e}_x + \sin\varphi\hat{e}_y$$

This represents the rate of variation of $\overline{r}$ with ρ keeping φ and z constant⇒ it is a vector parallel to the ρ -axis

$$\overline{e}_\varphi = \frac{\partial\overline{r}}{\partial\varphi} = \frac{\partial(\rho\cos\varphi\,\hat{e}_x + \rho\sin\varphi\,\hat{e}_y + z\,\hat{e}_z)}{\partial\varphi} = -\rho\sin\varphi\hat{e}_x + \rho\cos\varphi\hat{e}_y$$

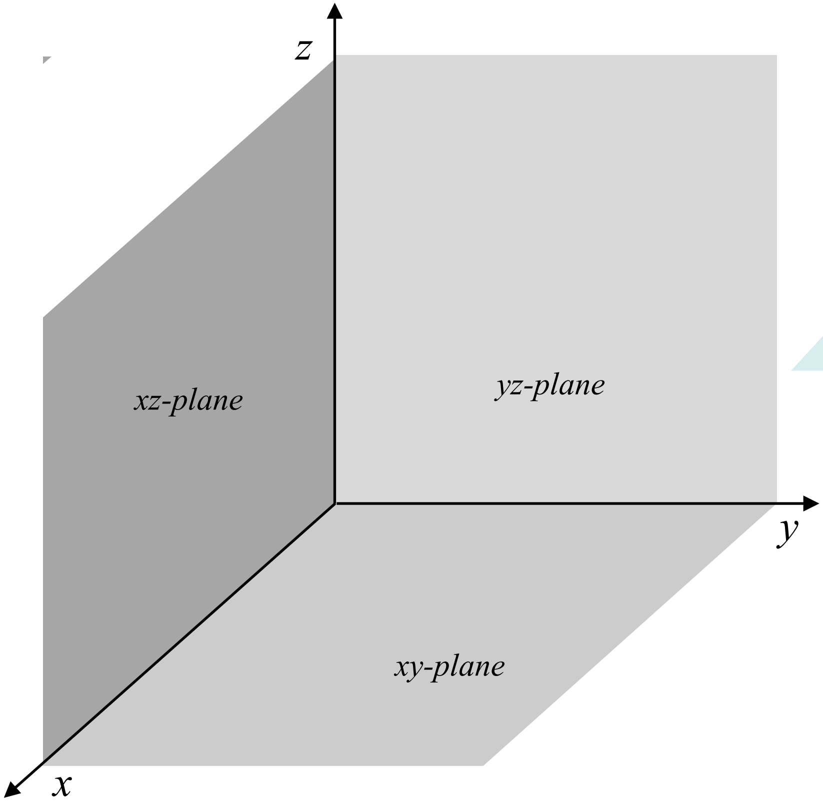
This represents the rate of variation of $\overline{r}$ with φ keeping ρ and z constant⇒ it is a vector parallel to the φ -axis

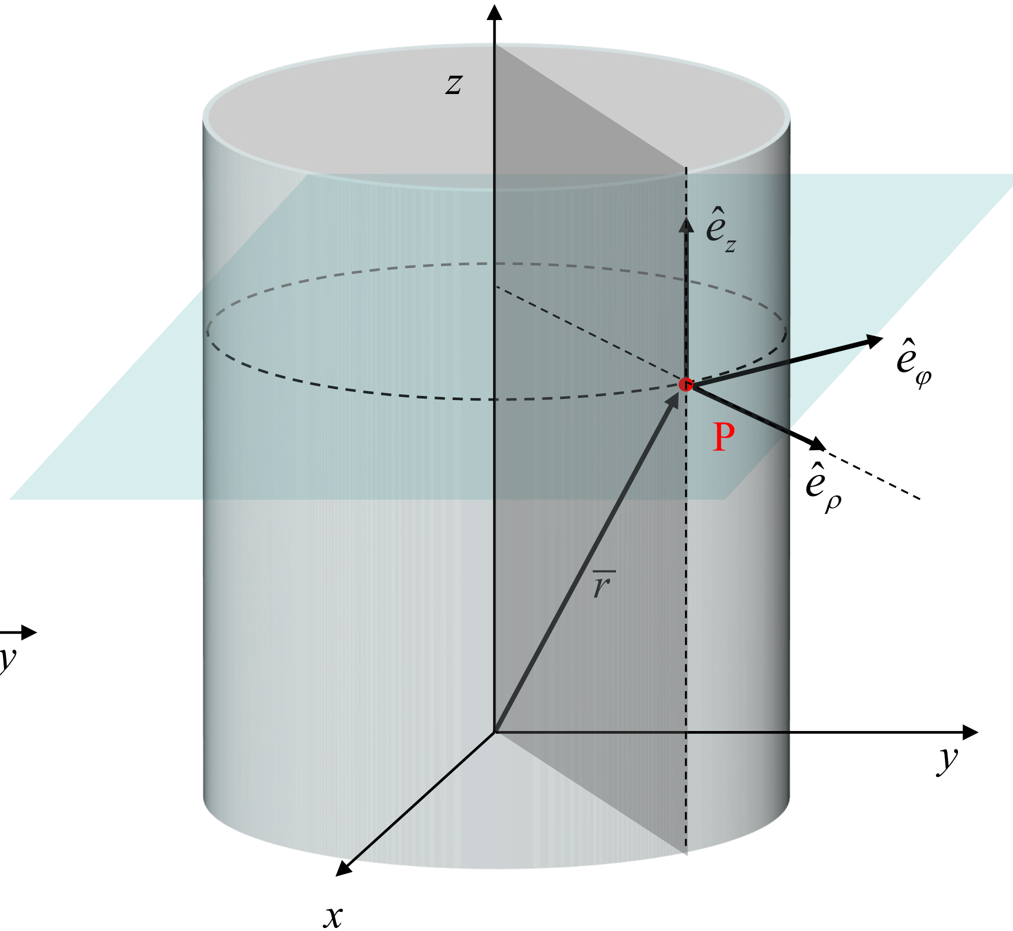
EXERCISE:
Express in a cylindrical coordinate system (i.e. using $\hat{e}_\rho$ and $\hat{e}_\varphi$) the vector (from the origin):

$$\overline{v} = x\hat{e}_x + y\hat{e}_y$$

COORDINATE CURVES

The coordinate curves are defined by the intersection of the coordinate surfaces





Cartesian coordinate system:

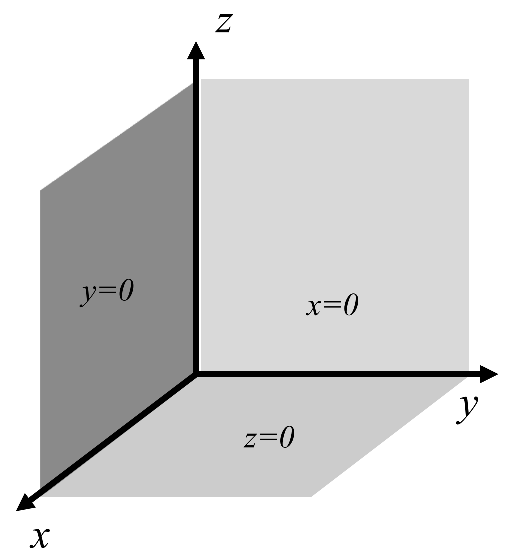
- coordinate surfaces: xy-plane yz-plane and xz-plane
- coordinate curves: x-axis, y-axis, z-axis.

Cylindrical coordinate system:

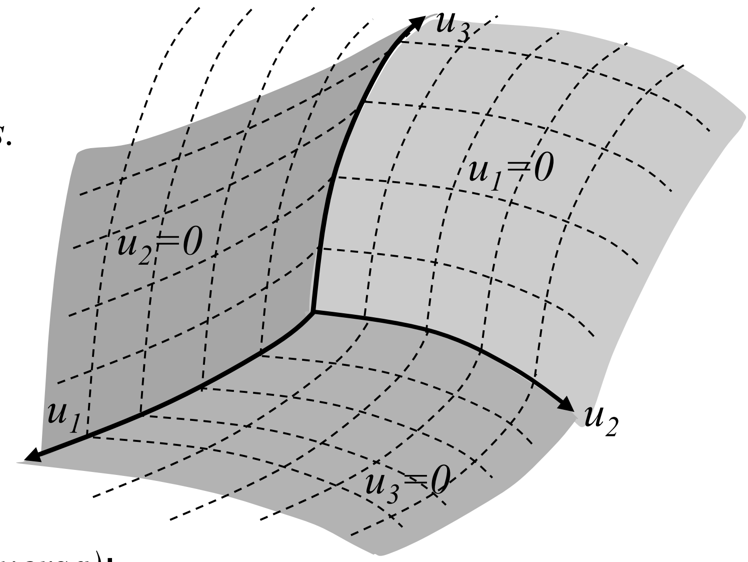
- coordinate surfaces: **a cylinder** with axis along the z-axis, **a plane** parallel to the xy-plane and **a plane** that contains the z-axis
- coordinate curves: **a straight line** parallel to the z-axis, **a circle** parallel to the xy-plane and centred on the z-axis, **a straight line** parallel to the xy-plane and passing by the z-axis

CURVILINEAR COORDINATE SYSTEMS

Consider a cartesian coordinate system *x, y, z*
The surfaces x=c   y=c and z=c are the coordinate surfaces.
Their intersection defines the x-axis, y-axis and z-axis



CURVILINEAR COORDINATE SYSTEMS



Consider a cartesian coordinate system x, y, z

The surfaces $x=c$ $y=c$ and $z=c$ are the coordinate surfaces.
Their intersection defines the x-axis, y-axis and z-axis

Consider another coordinate system defined by the variables u_1, u_2, u_3

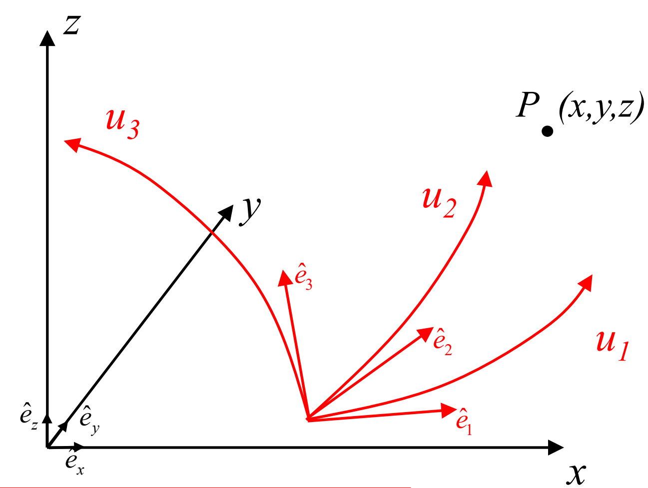
We assume that there is a one-to-one relationship between x_i and u_i,
so that x_i can be expressed as a function of u_i *(and vice-versa)*:

$$\begin{cases} x = f(u_1, u_2, u_3) \\ y = g(u_1, u_2, u_3) \\ z = h(u_1, u_2, u_3) \end{cases}$$

- The surfaces defined by $u_i=c$ are called coordinate surfaces
- The curves defined by the intersection of the coordinate surfaces are called coordinate curves
- The 3 curves u_1, u_2, u_3 are the coordinate axes

CURVILINEAR COORDINATES: basis and scale factors

	Point P	the basis is defined by the unit vectors:	$d\bar{r}$
CARTESIAN	x, y, z	$\hat{e}_x \quad \hat{e}_y \quad \hat{e}_z$	(dx, dy, dz)
CURVILINEAR	u_1, u_2, u_3	?	?



DEFINITION

An orthogonal curvilinear coordinate system has an orthonormal basis $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ in each point and

$$\hat{e}_i = \frac{1}{h_i}\frac{\partial \bar{r}}{\partial u_i} \quad with \quad scale\ factor \quad h_i = \left|\frac{\partial \bar{r}}{\partial u_i}\right|$$

Orthonormal:

- magnitude 1 $\quad |\hat{e}_i| = \left|\frac{1}{h_i}\frac{\partial \bar{r}}{\partial u_i}\right| = \frac{1}{|h_i|}\left|\frac{\partial \bar{r}}{\partial u_i}\right| = 1$
- orthogonal $\quad \hat{e}_i \cdot \hat{e}_j = \frac{1}{h_i}\frac{\partial \bar{r}}{\partial u_i} \cdot \frac{1}{h_j}\frac{\partial \bar{r}}{\partial u_j} = 0 \quad for\ i \neq j$

$$\Rightarrow \hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

Kronecker delta

$$d\bar{r} = \frac{\partial \bar{r}}{\partial u_1}du_1 + \frac{\partial \bar{r}}{\partial u_2}du_2 + \frac{\partial \bar{r}}{\partial u_3}du_3 = h_1\hat{e}_1du_1 + h_2\hat{e}_2du_2 + h_3\hat{e}_3du_3$$

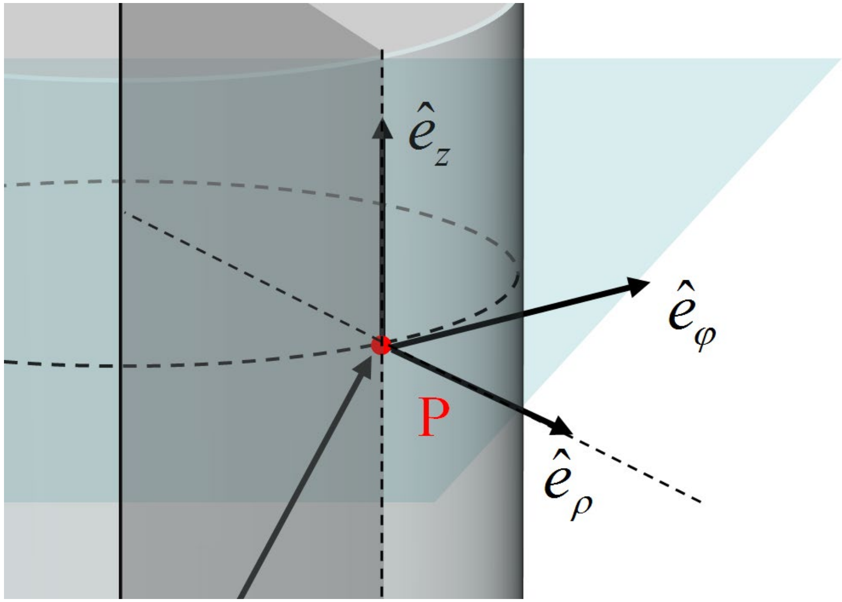
$$\boxed{d\bar{r} = \sum_i h_i du_i \hat{e}_i}$$

ORTHOGONAL CURVILINEAR COORDINATE SYSTEMS

Two coordinate curves u_i and u_j are orthogonal if their basis are orthogonal where the curves intersect:

$$\Rightarrow \frac{\partial \overline{r}}{\partial u_i} \cdot \frac{\partial \overline{r}}{\partial u_j} = 0 \qquad \text{with } i \neq j$$

example from the cylindrical coordinate system



If the coordinate system is orthogonal, scalar product and cross product can be calculated in the ”usual” way:

$$\left.\begin{array}{l} \overline{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3 \\ \overline{w} = w_3\hat{e}_3 + w_3\hat{e}_3 + w_3\hat{e}_3 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \overline{v}\cdot\overline{w} = v_1w_1 + v_2w_2 + v_3w_3 \\ \overline{v}\times\overline{w} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \end{array}\right.$$

Proof: try it at home. You can apply the same logic we have used in week 1 when we discussed ”basic vector algebra”

DIFFERENTIAL ELEMENTS

In a Cartesian coordinate system:

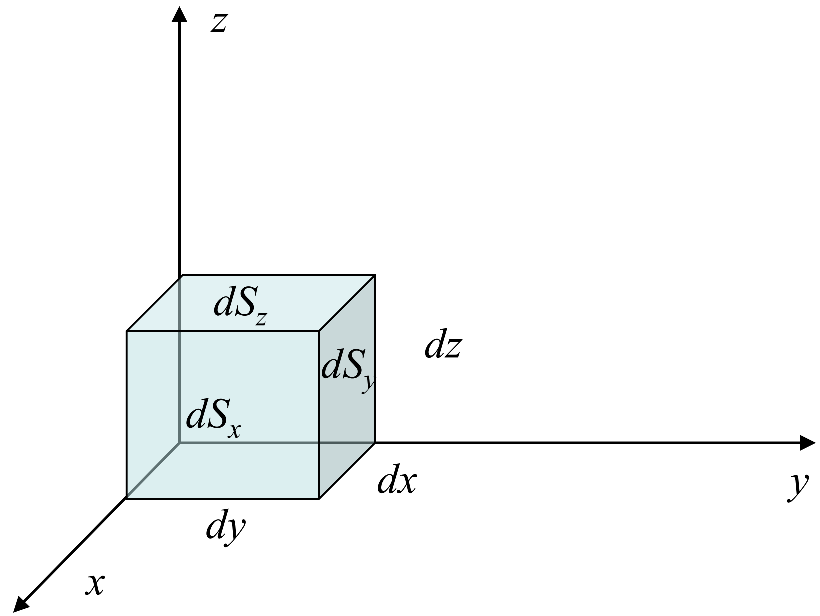
$$d\bar{r} = (dx, dy, dz) = dx\hat{e}_x + dy\hat{e}_y + dz\hat{e}_z$$

$$dS_x = dydz$$

$$dS_y = dxdz$$

$$dS_z = dxdy$$

$$dV = dxdydz$$



In an orthogonal curvilinear coordinate system:

$$d\bar{r} = h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3$$

$$dl_1 = h_1 du_1$$

$$dl_2 = h_2 du_2$$

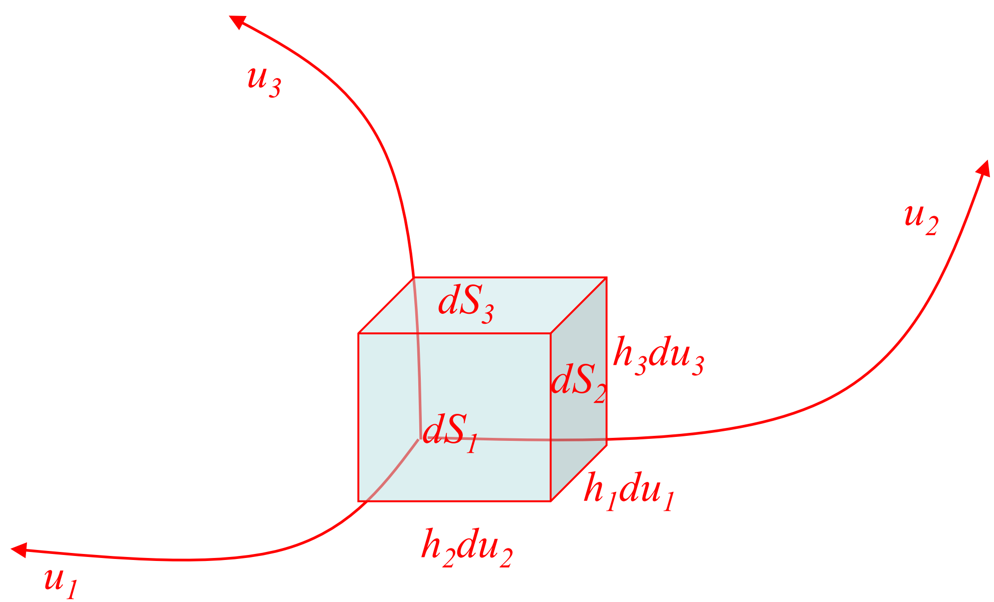
$$dl_3 = h_3 du_3$$

$$dS_1 = h_2 h_3 du_2 du_3$$

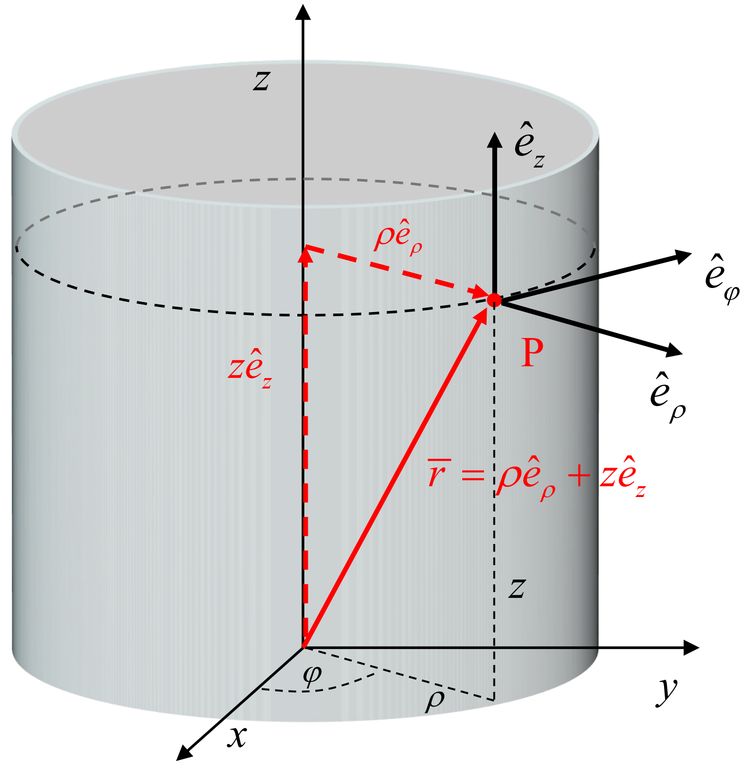
$$dS_2 = h_1 h_3 du_1 du_3$$

$$dS_3 = h_1 h_2 du_1 du_2$$

$$dV = h_1 h_2 h_3 du_1 du_2 du_3$$



CYLINDRICAL COORDINATE SYSTEM



Orthonormal basis

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases}$$

$$\overline{r} = \rho\cos\varphi\,\hat{e}_x + \rho\sin\varphi\,\hat{e}_y + z\,\hat{e}_z$$

$$\hat{e}_i = \frac{1}{h_i}\frac{\partial\overline{r}}{\partial u_i} \quad \textit{with} \quad h_i = \left|\frac{\partial\overline{r}}{\partial u_i}\right|$$

$$\begin{cases} \dfrac{\partial\overline{r}}{\partial\rho} = \cos\varphi\hat{e}_x + \sin\varphi\hat{e}_y \\ \dfrac{\partial\overline{r}}{\partial\varphi} = -\rho\sin\varphi\hat{e}_x + \rho\cos\varphi\hat{e}_y \\ \dfrac{\partial\overline{r}}{\partial z} = \hat{e}_z \end{cases}$$

$$\begin{cases} h_\rho = \sqrt{\cos^2\varphi + \sin^2\varphi} = 1 \\ h_\varphi = \sqrt{(-\rho\sin\varphi)^2 + (\rho\cos\varphi)^2} = \rho \\ h_z = 1 \end{cases}$$

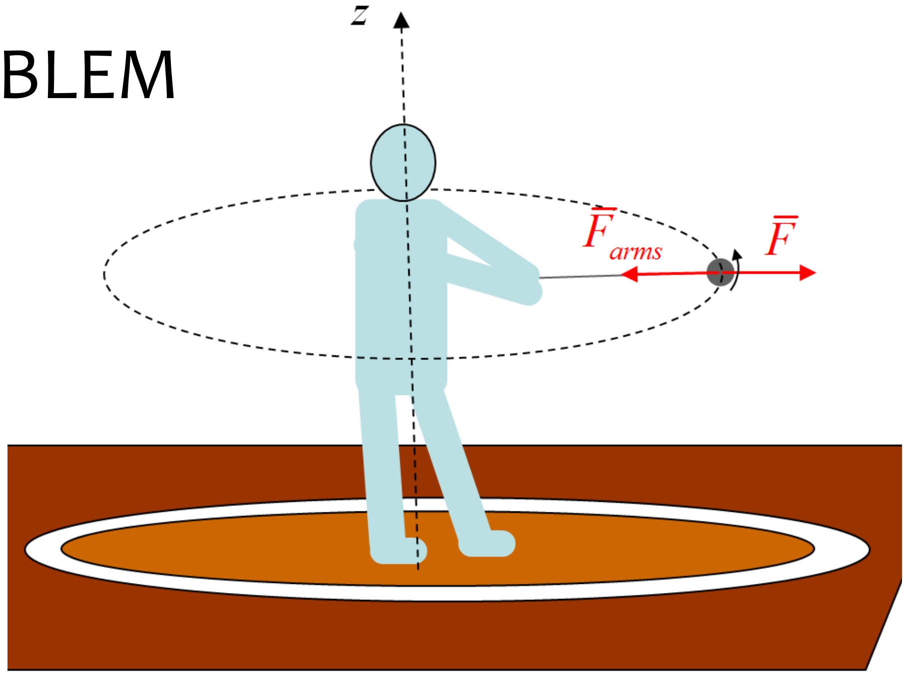
$$\begin{cases} \hat{e}_\rho = \dfrac{1}{h_\rho}\dfrac{\partial\overline{r}}{\partial\rho} = \cos\varphi\hat{e}_x + \sin\varphi\hat{e}_y \\ \hat{e}_\varphi = \dfrac{1}{h_\varphi}\dfrac{\partial\overline{r}}{\partial\varphi} = -\sin\varphi\hat{e}_x + \cos\varphi\hat{e}_y \\ \hat{e}_z = \dfrac{1}{h_z}\dfrac{\partial\overline{r}}{\partial z} = \hat{e}_z \end{cases}$$

$$\boxed{\begin{aligned} dl_z &= dz \\ dl_\rho &= d\rho \\ dl_\varphi &= \rho d\varphi \\ dS_\rho &= \rho d\varphi dz \\ dS_z &= \rho d\varphi d\rho \\ dV &= \rho d\rho d\varphi dz \end{aligned}}$$

EXERCISE: express in a cylindrical coordinate system the position vector $\overline{r}$ of a point located at coordinates ρ, φ, z.

TARGET PROBLEM

$$\overline{F}_{arms} = -\overline{F}$$
$$\overline{F} = m\overline{a}$$
$$\overline{a} = \frac{d\overline{v}}{dt} \equiv \dot{\overline{v}}$$
$$\overline{v} = \frac{d\overline{r}}{dt} \equiv \dot{\overline{r}}$$
$$\overline{r} = \rho_0 \hat{e}_\rho + z_0 \hat{e}_z \qquad \textit{(in cylindrical coord.)}$$



We assume ρ_0 and z_0 constant

$$\overline{F}_{arms} = -m\overline{a} = -m\dot{\overline{v}} = -m\ddot{\overline{r}}$$

$$\ddot{\overline{r}} = \frac{d}{dt}\dot{\overline{r}} = \frac{d}{dt}\left(\underbrace{\dot{\rho}_0}_{=0 \text{ (the radius does not change in time)}}\hat{e}_\rho + \rho_0 \dot{\hat{e}}_\rho + \underbrace{\dot{z}_0}_{=0 \text{ (the rotation is on a plane } z=\text{constant)}}\hat{e}_z + z_0 \dot{\hat{e}}_z\right) =$$

$$\dot{\hat{e}}_\rho = \frac{d}{dt}\left(\cos\varphi \hat{e}_x + \sin\varphi \hat{e}_y\right) = -\dot{\varphi}\sin\varphi \hat{e}_x + \dot{\varphi}\cos\varphi \hat{e}_y = \dot{\varphi}\left(-\sin\varphi \hat{e}_x + \cos\varphi \hat{e}_y\right) = \dot{\varphi}\hat{e}_\varphi$$
$$\dot{\hat{e}}_z = \frac{d}{dt}(0,0,1) = 0$$

$$= \frac{d}{dt}\left(\rho_0 \dot{\varphi}\hat{e}_\varphi\right) = \left(\dot{\rho}_0 \dot{\varphi}\hat{e}_\varphi + \rho_0 \ddot{\varphi}\hat{e}_\varphi + \rho_0 \dot{\varphi}\dot{\hat{e}}_\varphi\right) = \left(\dot{\rho}_0 \dot{\varphi}\hat{e}_\varphi + \rho_0 \ddot{\varphi}\hat{e}_\varphi - \rho_0 \dot{\varphi}\dot{\varphi}\hat{e}_\rho\right) = \rho_0\left(\ddot{\varphi}\hat{e}_\varphi - \dot{\varphi}^2 \hat{e}_\rho\right)$$

$$\dot{\hat{e}}_\varphi = \frac{d}{dt}\left(-\sin\varphi \hat{e}_x + \cos\varphi \hat{e}_y\right) = -\dot{\varphi}\cos\varphi \hat{e}_x - \dot{\varphi}\sin\varphi \hat{e}_y = -\dot{\varphi}\left(\cos\varphi \hat{e}_x + \sin\varphi \hat{e}_y\right) = -\dot{\varphi}\hat{e}_\rho$$

$$\overline{F}_{arms} = -m\ddot{\overline{r}} = m\rho_0\left(\dot{\varphi}^2 \hat{e}_\rho - \ddot{\varphi}\hat{e}_\varphi\right)$$

In the book, this is the target problem of Chapter 3.

PRACTICAL EXAMPLE: **THE BIOT-SAVART LAW**

The magnetic field in a point P of a steady line current is given by the Biot-Savart law:

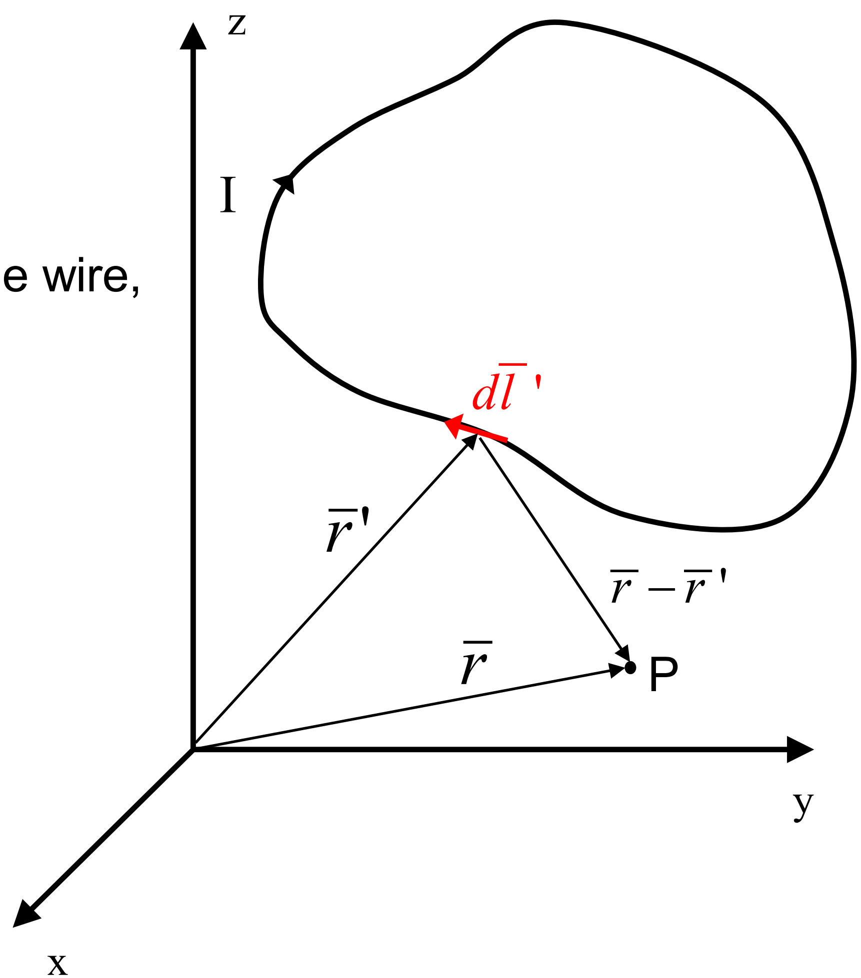
$$\overline{B}(\overline{\mathrm{r}}) = \frac{\mu_0 I}{4\pi} \int_L \frac{d\overline{l}\,' \times (\overline{r} - \overline{r}\,')}{\left|\overline{r} - \overline{r}\,'\right|^3}$$

Where $d\overline{l}\,'$ is an infinitesimal length along the wire,

$\overline{r}$ is the position vector of the point P and

$\overline{r}\,'$ is a vector from the origin to $d\overline{l}\,'$

Therefore, $\overline{r} - \overline{r}\,'$ is a vector from $d\overline{l}\,'$ to P



PRACTICAL EXAMPLE: **THE BIOT-SAVART LAW**

(see section 8.2 in "Teoretisk Elektroteknik Stationära fenomen" by G. Petersson)

The magnetic field in a point P of a steady line current is given by the Biot-Savart law:

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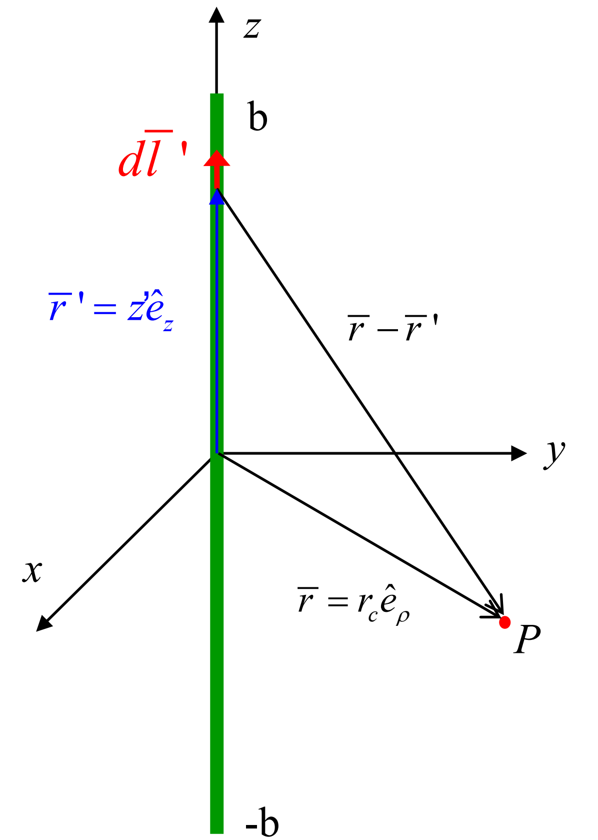


Calculate the magnetic field in P: $x_0\hat{e}_x + y_0\hat{e}_y$ produced by a straight wire along the z-axis with current I and length 2b and centred at z=0.

SOLUTION:

If $r_c = \sqrt{x_0^2 + y_0^2}$ *is the distance from the origin to P,*

in a cylindrical coordinate system:

$$\begin{cases} \overline{r} = r_c\hat{e}_\rho + z\hat{e}_z = r_c\hat{e}_\rho & \text{(P is at z=0)} \\ \overline{r}\,' = z'\hat{e}_z & \text{(the wire is along the z-axis)} \\ \overline{r} - \overline{r}\,' = r_c\hat{e}_\rho - z'\hat{e}_z & \Rightarrow\ |\overline{r} - \overline{r}\,'| = \sqrt{r_c^2 + z'^2} \end{cases}$$

The curve L (a wire along the z-axis) can be parameterized as:

$$\begin{cases} \overline{l}(z') = z'\hat{e}_z & \Rightarrow d\overline{l}\,' = dz'\hat{e}_z \\ \text{with} \quad z' : -b \to +b \end{cases}$$

$$d\overline{l}\,' \times (\overline{r} - \overline{r}\,') = dz'\hat{e}_z \times (r_c\hat{e}_\rho - z'\hat{e}_z) = r_c dz'\hat{e}_\varphi$$

$$\overline{B}(\overline{r}) = \frac{\mu_0 I}{4\pi}\int_{-b}^{b} \frac{r_c dz'\hat{e}_\varphi}{(r_c^2 + z'^2)^{3/2}} = \hat{e}_\varphi \frac{\mu_0 I}{4\pi} r_c \int_{-b}^{b} \frac{dz'}{(r_c^2 + z'^2)^{3/2}} = \hat{e}_\varphi \frac{\mu_0 I}{4\pi} r_c \left[\frac{z'}{r_c^2\sqrt{r_c^2 + z'^2}}\right]_{-b}^{b} = \frac{\mu_0 I}{4\pi r_c}\frac{2b}{\sqrt{r_c^2 + b^2}}\hat{e}_\varphi$$

$\hat{e}_\varphi$ does not depend on z, so you can move it out from the integration.
But, in general, $\hat{e}_\varphi$ is not constant so you need to integrate it.

If the wire is infinitely long, we can calculate the limit for b→∞ $\Rightarrow \ \overline{B}(\overline{r}) = \dfrac{\mu_0 I}{2\pi r_c}\hat{e}_\varphi$

WHICH STATEMENT IS WRONG?

1- $\int z\hat{e}_\rho dz = \hat{e}_\rho \int z dz$

2- $\hat{e}_z$ **is constant everywhere**

3- $\int \varphi \hat{e}_\varphi d\varphi = \hat{e}_\varphi \int \varphi d\varphi$

4- **In a cylindrical coordinate system**

the position vector is $\bar{r} = \rho \hat{e}_\rho + z\hat{e}_z$

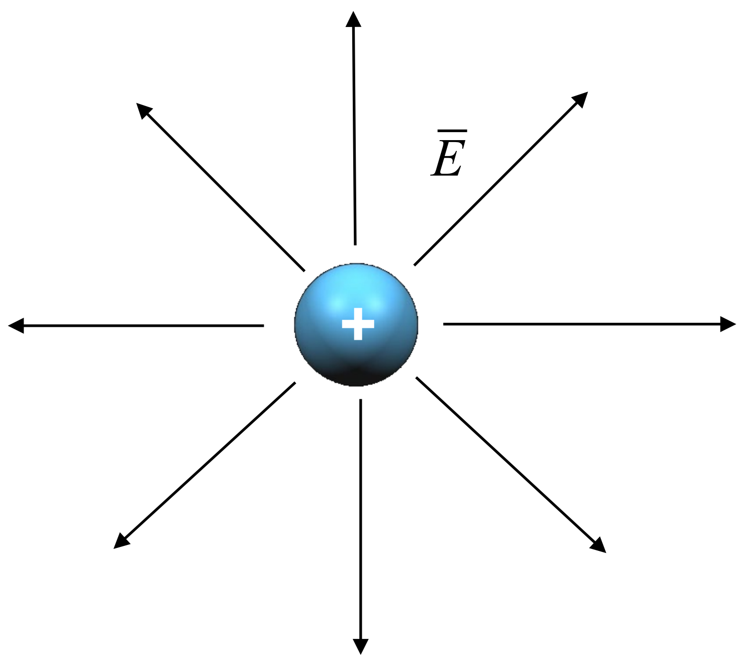
TARGET PROBLEM

The electric field is conservative. The electrostatic potential V is defined by: $\bar{E} = -gradV$

$$\left.\begin{array}{r} \bar{E} = -gradV \\ div\bar{E} = 0 \end{array}\right\} \Longrightarrow div\left(gradV\right) = \boxed{\nabla^2 V = 0}$$ Laplace's equation

first Maxwell's equation with no charge

Calculate the electrostatic potential generated outside a spherical charge.



Due to the spherical symmetry, the solution will depend only on the radius: $V = V(r)$

with $r = |\bar{r}|$

We need to:

- introduce spherical coordinates
- calculate gradient and divergence in spherical coordinates
- solve the equation

SPHERICAL COORDINATES

spherical coordinates are an
example of curvilinear coordinates

cartesian coord.

$P: \; x_0, \; y_0, \; z_0$

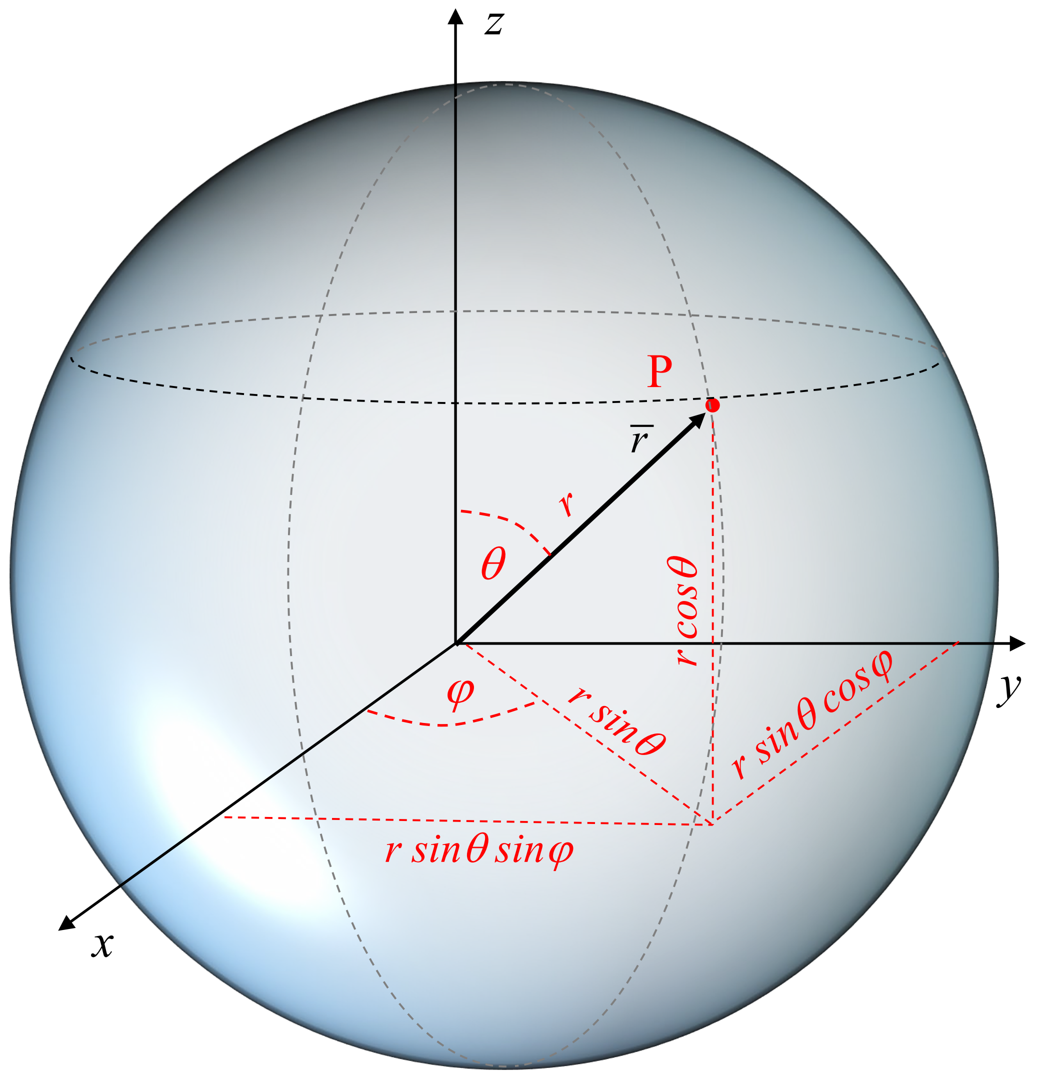
spherical coord.

$P: \; r, \; \theta, \; \varphi$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \tan\theta = \dfrac{\sqrt{x^2 + y^2}}{z} \\ \tan\varphi = y / x \end{cases}$$

$$0 \le r \le \infty$$
$$0 \le \theta \le \pi$$
$$0 \le \varphi \le 2\pi$$

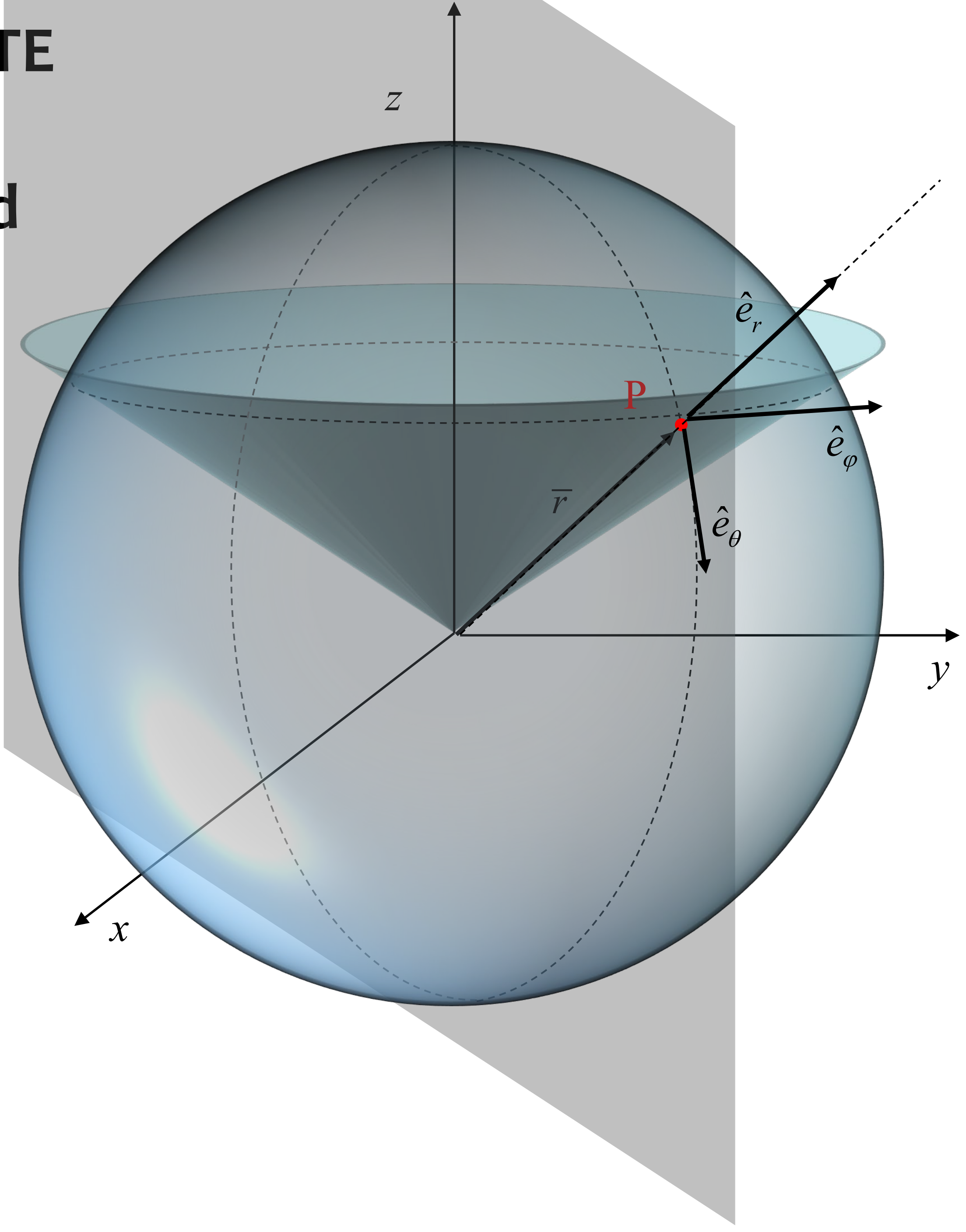
$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$



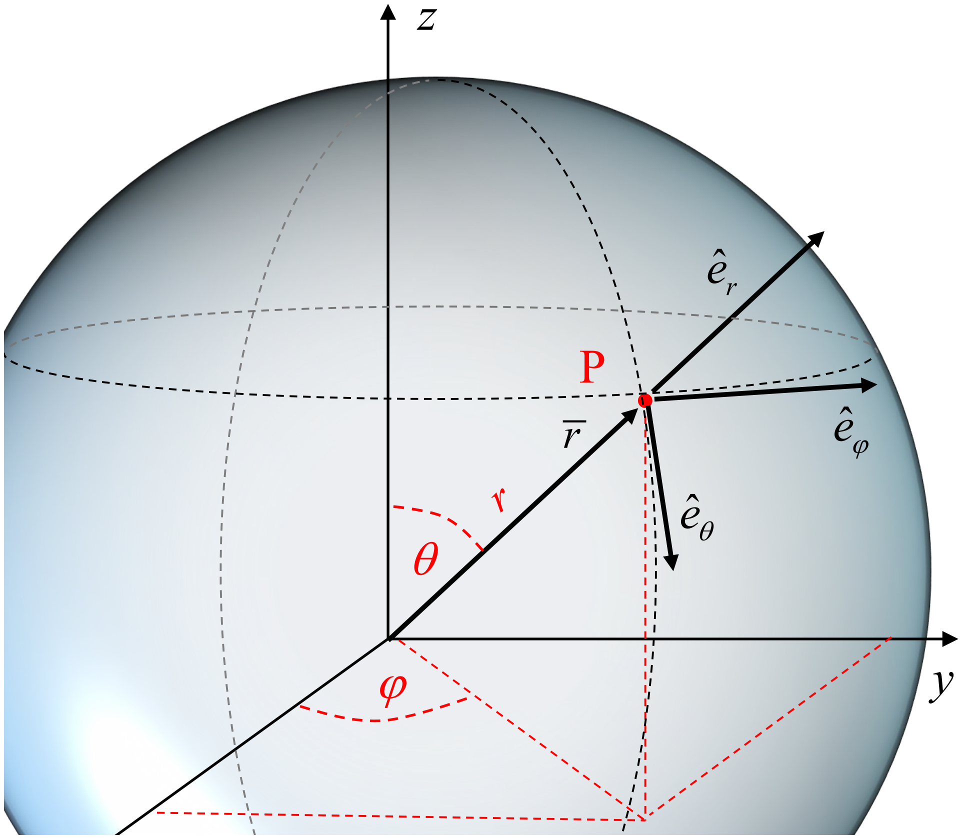
SPHERICAL COORDINATE Systems: coordinate surfaces and coordinate curves

- coordinate surfaces:
  - **a sphere** centred in the origin
  - **a cone** with axis along the z-axis and vertex centred in the origin
  - **a plane** that contains the z-axis

- coordinate curves:
  - **a straight line** from the origin to P
  - **a circle** parallel to the xy-plane and with centre on the z-axis
  - **a circle** parallel the z-axis and centred in the origin



SPHERICAL COORDINATE SYSTEM



Orthonormal basis

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

$$\overline{r} = r\sin\theta\cos\varphi\,\hat{e}_x + r\sin\theta\sin\varphi\,\hat{e}_y + r\cos\theta\,\hat{e}_z$$

$$\hat{e}_i = \frac{1}{h_i}\frac{\partial\overline{r}}{\partial u_i} \quad \textit{with} \quad h_i = \left|\frac{\partial\overline{r}}{\partial u_i}\right|$$

$$\begin{cases} \dfrac{\partial\overline{r}}{\partial r} = \sin\theta\cos\varphi\hat{e}_x + \sin\theta\sin\varphi\hat{e}_y + \cos\theta\hat{e}_z \\ \dfrac{\partial\overline{r}}{\partial\theta} = r\cos\theta\cos\varphi\hat{e}_x + r\cos\theta\sin\varphi\hat{e}_y - r\sin\theta\hat{e}_z \\ \dfrac{\partial\overline{r}}{\partial\varphi} = -r\sin\theta\sin\varphi\hat{e}_x + r\sin\theta\cos\varphi\hat{e}_y \end{cases}$$

$$\begin{cases} h_r = \sqrt{\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta} = 1 \\ h_\theta = \sqrt{(r\cos\theta\cos\varphi)^2 + (r\cos\theta\sin\varphi)^2 + (r\sin\theta)^2} = r \\ h_\varphi = \sqrt{(r\sin\theta\sin\varphi)^2 + (r\sin\theta\cos\varphi)^2} = r\sin\theta \end{cases}$$

$$\Rightarrow \begin{cases} \hat{e}_r = \dfrac{1}{h_r}\dfrac{\partial\overline{r}}{\partial r} = \sin\theta\cos\varphi\hat{e}_x + \sin\theta\sin\varphi\hat{e}_y + \cos\theta\hat{e}_z \\ \hat{e}_\theta = \dfrac{1}{h_\theta}\dfrac{\partial\overline{r}}{\partial\theta} = \cos\theta\cos\varphi\hat{e}_x + \cos\theta\sin\varphi\hat{e}_y - \sin\theta\hat{e}_z \\ \hat{e}_\varphi = \dfrac{1}{h_\varphi}\dfrac{\partial\overline{r}}{\partial\varphi} = -\sin\varphi\hat{e}_x + \cos\varphi\hat{e}_y \end{cases}$$

$$\boxed{\begin{aligned} dS_r &= r^2\sin\theta d\theta d\varphi \\ dV &= r^2\sin\theta dr d\theta d\varphi \end{aligned}}$$

EXERCISE: *express* $\overline{r}$ *using* $\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi$

GRADIENT IN CURVILINEAR COORDINATE SYSTEMS

In a Cartesian coordinate system: $$grad\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = \frac{\partial\phi}{\partial x}\hat{e}_x + \frac{\partial\phi}{\partial y}\hat{e}_y + \frac{\partial\phi}{\partial z}\hat{e}_z$$

And in a curvilinear coordinate system?
We must express $grad\phi$ in terms of the curvilinear basis $\hat{e}_1, \hat{e}_2, \hat{e}_3$:

$$grad\phi = f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3$$

since $d\phi = grad\phi \cdot d\bar{r}$ and $d\bar{r} = h_1 du_1\hat{e}_1 + h_2 du_2\hat{e}_2 + h_3 du_3\hat{e}_3$

$$d\phi = \left(f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3\right)\cdot\left(h_1 du_1\hat{e}_1 + h_2 du_2\hat{e}_2 + h_3 du_3\hat{e}_3\right) = f_1 h_1 du_1 + f_2 h_2 du_2 + f_3 h_3 du_3$$

But also, writing ϕ as a function of u_i: $d\phi = \frac{\partial\phi}{\partial u_1}du_1 + \frac{\partial\phi}{\partial u_2}du_2 + \frac{\partial\phi}{\partial u_3}du_3$

Therefore: $$f_1 = \frac{1}{h_1}\frac{\partial\phi}{\partial u_1}, \quad f_2 = \frac{1}{h_2}\frac{\partial\phi}{\partial u_2}, \quad f_3 = \frac{1}{h_3}\frac{\partial\phi}{\partial u_3}$$

$$\boxed{grad\phi = \sum_i \frac{1}{h_i}\frac{\partial\phi}{\partial u_i}\hat{e}_i}$$

GRADIENT IN CURVILINEAR COORDINATE SYSTEMS

THE GRADIENT

- in a cylindrical coordinate system:

$$grad\,\phi = \frac{1}{h_\rho}\frac{\partial \phi}{\partial \rho}\hat{e}_\rho + \frac{1}{h_\varphi}\frac{\partial \phi}{\partial \varphi}\hat{e}_\varphi + \frac{1}{h_z}\frac{\partial \phi}{\partial z}\hat{e}_z = \boxed{\frac{\partial \phi}{\partial \rho}\hat{e}_\rho + \frac{1}{\rho}\frac{\partial \phi}{\partial \varphi}\hat{e}_\varphi + \frac{\partial \phi}{\partial z}\hat{e}_z}$$

- in a spherical coordinate system:

$$grad\,\phi = \frac{1}{h_r}\frac{\partial \phi}{\partial r}\hat{e}_r + \frac{1}{h_\theta}\frac{\partial \phi}{\partial \theta}\hat{e}_\theta + \frac{1}{h_\varphi}\frac{\partial \phi}{\partial \varphi}\hat{e}_\varphi = \boxed{\frac{\partial \phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial \phi}{\partial \varphi}\hat{e}_\varphi}$$

EXERCISE: *calculate in a spherical coordinate system:* $grad\left(\frac{1}{r}\right)$

DIVERGENCE IN CURVILINEAR COORDINATE SYSTEMS

$$\boxed{div\bar{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(A_1h_2h_3) + \frac{\partial}{\partial u_2}(A_2h_3h_1) + \frac{\partial}{\partial u_3}(A_3h_1h_2)\right]}$$

Proof: see theorem 10.2, page 186

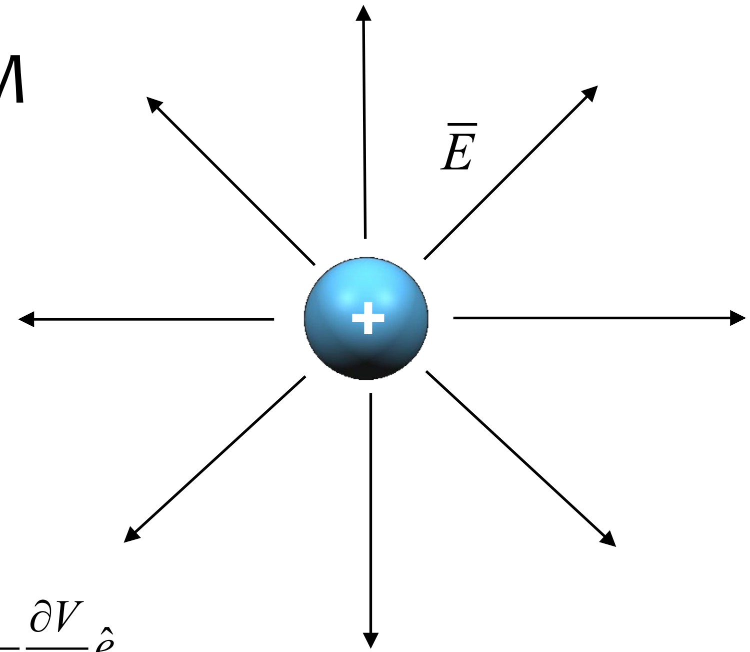
EXERCISE: *express* $div\bar{A}$ *in a spherical coord. sys.*

CURL IN CURVILINEAR COORDINATE SYSTEMS

$$\boxed{rot\bar{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}}$$

Proof: see theorem 10.3, page 188

TARGET PROBLEM



Due to spherical symmetry

$$V = V(r)$$

$$\text{with} \quad r = |\overline{r}|$$

$$\nabla^2 V = 0$$

Which can be written as:

$$\nabla^2 V = div\left(gradV\right) = 0$$

Due to spherical symmetry, the solution is easy in spherical coordinates

$$gradV = \left(\frac{\partial V}{\partial r}, \frac{1}{r}\frac{\partial V}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}\right) = \frac{\partial V}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}\hat{e}_\varphi$$

$$div\overline{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(A_1 h_2 h_3\right) + \frac{\partial}{\partial u_2}\left(A_2 h_3 h_1\right) + \frac{\partial}{\partial u_3}\left(A_3 h_1 h_2\right)\right] = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(A_r r^2\sin\theta\right) + \frac{\partial}{\partial \theta}\left(A_\theta r\sin\theta\right) + \frac{\partial}{\partial \varphi}\left(rA_\varphi\right)\right]$$

$$div\left(gradV\right) = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(\frac{\partial V}{\partial r}r^2\sin\theta\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\underbrace{\frac{\partial V}{\partial \theta}}_{=0}r\sin\theta\right) + \frac{\partial}{\partial \varphi}\left(r\frac{1}{r\sin\theta}\underbrace{\frac{\partial V}{\partial \varphi}}_{=0}\right)\right] = \frac{2}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2}$$

Spherical symmetry $\Rightarrow$ *no* θ *and no* φ *dependence*

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} = 0 \quad \Rightarrow \quad V(r) = -\frac{c}{r} + d \qquad \overline{E} = -gradV = \frac{c}{r^2}\hat{e}_r$$

WHICH STATEMENT IS WRONG?

1- The scale factor is necessary to calculate the gradient

2- The scale factor is necessary to calculate the divergence

3- A spherical coordinate system is an example of an orthogonal coordinate system

4- In a spherical coordinate system

the position vector is: $\overline{r} = r\hat{e}_r + \theta\hat{e}_\theta + \varphi\hat{e}_\varphi$