VEKTORANALYS

HT 2021

CELTE / CENMI

ED1110

SOME SPECIAL VECTOR FIELDS
AND
LAPLACE AND POISSON EQUATIONS

ÖVNINGAR

Kursvecka 6
Kapitel 16
Avsnitt: målproblem, 17.1, 17.2, 17.3
Avsnitt 17.5 (inte bevisen)

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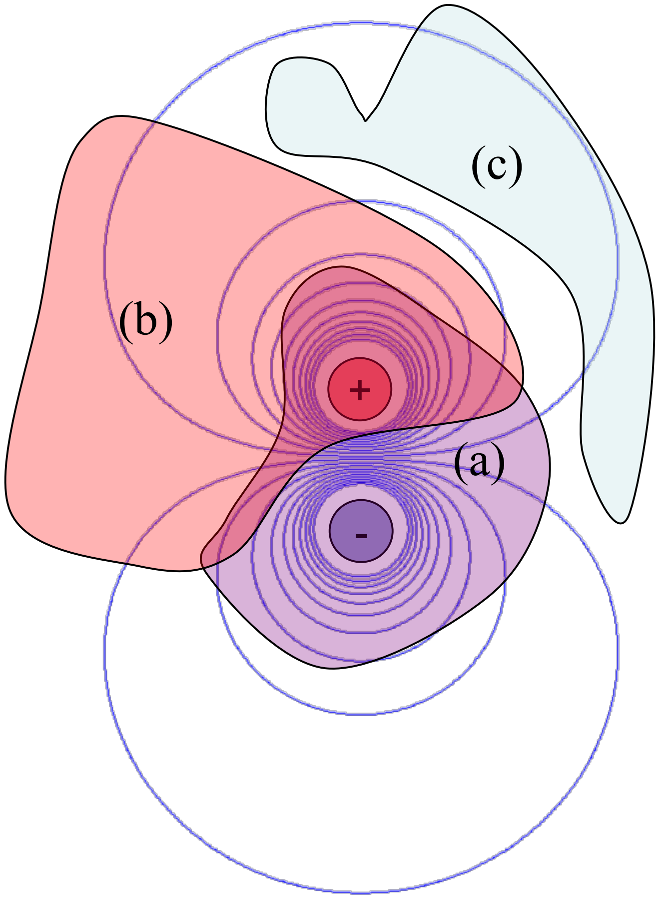
PROBLEM 1

A dipole is formed by two point sources with charge +c and -c
Calculate the flux of the dipole field on a closed surface S that

(a) encloses both poles
(b) encloses only the plus pole
(c) does not enclose any pole



SOLUTION

Vector field from a point source located in $\overline{r}_0$: $\quad \overline{A}(\overline{r}) = c\dfrac{\overline{r}-\overline{r}_0}{|\overline{r}-\overline{r}_0|^3}$

Vector field from two point sources with opposite charge located in $\overline{r}_1$ and $\overline{r}_2$ $\quad \overline{A}(\overline{r}) = c\dfrac{\overline{r}-\overline{r}_1}{|\overline{r}-\overline{r}_1|^3} - c\dfrac{\overline{r}-\overline{r}_2}{|\overline{r}-\overline{r}_2|^3}$

Flux from a point source: $\quad = \begin{cases} 0 & \textit{If the origin is outside } V \\ 4\pi c & \textit{If the origin is inside } V \end{cases}$

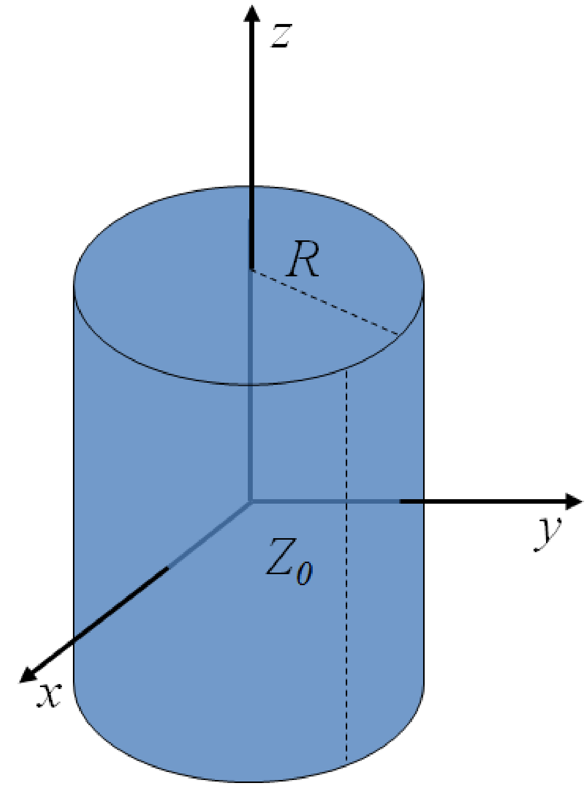
$$\oiint_S \left(c\frac{\overline{r}-\overline{r}_1}{|\overline{r}-\overline{r}_1|^3} - c\frac{\overline{r}-\overline{r}_2}{|\overline{r}-\overline{r}_2|^3} \right)\cdot d\overline{S} = \oiint_S c\frac{\overline{r}-\overline{r}_1}{|\overline{r}-\overline{r}_1|^3}\cdot d\overline{S} - \oiint_s c\frac{\overline{r}-\overline{r}_2}{|\overline{r}-\overline{r}_2|^3}\cdot d\overline{S}$$

(a)	$\oiint_S (\ )\cdot d\overline{S} =$	$4\pi c$	$-4\pi c$	$=0$
(b)	$\oiint_S (\ )\cdot d\overline{S} =$	$4\pi c$	0	$= 4\pi c$
(c)	$\oiint_S (\ )\cdot d\overline{S} =$	0	0	$= 0$

PROBLEM 2

Betrakta följande vektorfält (i sfäriska koordinater):

$$\overline{A} = \frac{3 + r^4 \sin\theta}{r^2}\hat{e}_r + r^2\hat{e}_\varphi$$



Beräkna flödet av vektorfältet ut genom en cylinderyta S med radien R, axel längs $\hat{e}_z$. längd z_o och centrum i origo. Flödet genom cylinderns två cirkulära lock skall medräknas (S är en sluten yta).

SOLUTION to problem 2

$$\int_s \bar{A}\cdot d\bar{S} = \int_s \left(\frac{3+r^4\sin\theta}{r^2}\hat{e}_r + r^2\hat{e}_\varphi \right)\cdot d\bar{S} = \int_s \left(\frac{3}{r^2}\hat{e}_r + r^2\sin\theta\hat{e}_r + r^2\hat{e}_\varphi \right)\cdot d\bar{S} =$$

$$= \underbrace{\int_s \left(\frac{3}{r^2}\hat{e}_r \right)\cdot d\bar{S}}_{\text{Theorem 16.1}} + \overbrace{\int_s \left(r^2\sin\theta\hat{e}_r + r^2\hat{e}_\varphi \right)\cdot d\bar{S}}^{\text{Gauss' theorem}} = \underbrace{12\pi}_{\text{Theorem 16.1}} + \overbrace{\int_V \nabla\cdot\left(r^2\sin\theta\hat{e}_r + r^2\hat{e}_\varphi \right) dV}^{\text{Gauss' theorem}} =$$

$$\left\{ \begin{array}{l} \text{in a spherical coordinate system: } \nabla\cdot\bar{v} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 v_r\right) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta v_\theta\right) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\varphi}\left(v_\varphi\right) \\ \text{so: } \nabla\cdot\left(r^2\sin\theta\hat{e}_r + r^2\hat{e}_\varphi \right) = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^4\sin\theta\right) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\varphi}\left(r^2\right) = \dfrac{4r^3\sin\theta}{r^2} = 4r\sin\theta \end{array} \right\}$$

$$= 12\pi + \int_V 4r\sin\theta dV =$$

$\{$Note that the volume is a cylinder. And that $r\sin\theta = \rho\}$

$$= 12\pi + \int_V 4\rho dV = 12\pi + \int_V 4\rho^2 d\varphi d\rho dz = 12\pi + 8\pi Z_0 \int_V \rho^2 d\rho = 12\pi + 8\pi Z_0 \left[\frac{1}{3}\rho^3 \right]_0^R = 12\pi + \frac{8}{3}\pi Z_0 R^3$$

SCALAR LAPLACIAN: USEFUL EXPRESSIONS

The Laplace equation is $\nabla^2 \phi = 0$

The Laplace operator ∇^2 is the divergence of the gradient:

$$\boxed{\nabla^2 \phi \equiv \nabla \cdot \nabla \phi}$$

CARTESIAN COORDINATE SYSTEM

- GRADIENT $\nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)$

- DIVERGENCE $\nabla \cdot \overline{A} = \frac{\partial A_z}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

CARTESIAN

$$\boxed{\nabla^2 \phi = \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right)}$$

CYLINDRICAL COORDINATE SYSTEM

- GRADIENT $\nabla \phi = \left(\frac{\partial \phi}{\partial \rho}, \frac{1}{\rho} \frac{\partial \phi}{\partial \varphi}, \frac{\partial \phi}{\partial z} \right)$

- DIVERGENCE $\nabla \cdot \overline{A} = \left(\frac{1}{\rho} \frac{\partial \left(\rho A_\rho \right)}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z} \right)$

CYLINDRICAL

$$\boxed{\nabla^2 \phi = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \phi}{\partial \varphi^2} + \frac{\partial^2 \phi}{\partial z^2}}$$

SPHERICAL COORDINATE SYSTEM

- GRADIENT $\nabla \phi = \left(\frac{\partial \phi}{\partial r}, \frac{1}{r} \frac{\partial \phi}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \varphi} \right)$

- DIVERGENCE $\nabla \cdot \overline{A} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 A_r \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta A_\theta \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \left(A_\varphi \right)$

SPHERICAL

$$\boxed{\nabla^2 \phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \varphi^2}}$$

PROBLEM 3

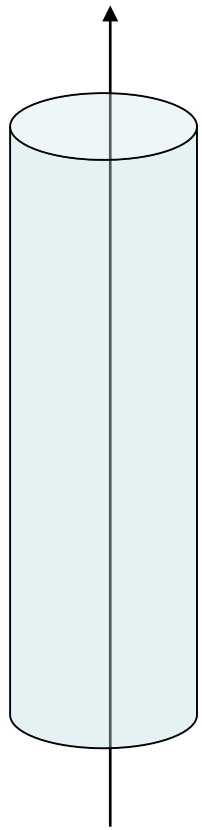
An infinitely long cylinder with radius R has a charge density ρ_c.
Calculate the potential and the electric field:

(a) Inside the cylinder
(b) Outside the cylinder

Assume the potential on the surface is V_o.

SOLUTION

Due to the symmetry of the problem,
the solution will depend on the radius only: $\phi=\phi(\rho)$



Inside the cylinder

$$\nabla^2\phi = -\frac{\rho_c}{\varepsilon_0}$$

cylindrical coord.

$$\nabla^2\phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \underbrace{\frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2}}_{=0} + \underbrace{\frac{\partial^2\phi}{\partial z^2}}_{=0} = -\frac{\rho_c}{\varepsilon_0}$$

Because the solution depends only on ρ

The equation becomes:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) = -\frac{\rho_c}{\varepsilon_0}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right)=-\frac{\rho_c}{\varepsilon_0} \xRightarrow{\text{Multiplying by }\rho} \frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right)=-\frac{\rho_c}{\varepsilon_0}\rho \xRightarrow{\text{Integrating in }\rho} \rho\frac{\partial\phi}{\partial\rho}=-\frac{\rho_c}{2\varepsilon_0}\rho^2+a \xRightarrow{\text{Dividing by }\rho} \frac{\partial\phi}{\partial\rho}=-\frac{\rho_c}{2\varepsilon_0}\rho+\frac{a}{\rho} \Rightarrow$$

$$\left[\bar{E}=-\nabla\phi=-\left(\frac{d\phi(\rho)}{d\rho},\frac{1}{\rho}\underbrace{\frac{d\phi(\rho)}{d\varphi}}_{=0},\underbrace{\frac{d\phi(\rho)}{dz}}_{=0}\right) \Rightarrow E_r=-\frac{d\phi(\rho)}{d\rho}=\boxed{+\frac{\rho_c}{2\varepsilon_0}\rho}-\frac{a}{\rho}\right]$$

Because the solution depends only on ρ

Divergent at ρ=0
A solution with a≠0 would not have a physical meaning
⇒ a=0

$$\frac{\partial\phi}{\partial\rho}=-\frac{\rho_c}{2\varepsilon_0}\rho \xRightarrow{\text{Integrating in }\rho} \boxed{\phi(\rho)=-\frac{\rho_c}{4\varepsilon_0}\rho^2+b}$$

Outside the cylinder

$\nabla^2\phi=0$ ← *There is no charge*

And the equation becomes:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right)=0 \xRightarrow{\text{Multiplying by }\rho} \frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right)=0 \xRightarrow{\text{Integrating in }\rho} \rho\frac{\partial\phi}{\partial\rho}=c \xRightarrow{\text{Dividing by }\rho} \frac{\partial\phi}{\partial\rho}=\frac{c}{\rho} \xRightarrow{\text{Integrating in }\rho} \boxed{\phi(\rho)=c\ln\rho+d}$$

$$E_r=-\frac{d\phi(\rho)}{d\rho}=\boxed{-\frac{c}{\rho}}$$

$$E_r^{in}(\rho) = +\frac{\rho_c}{2\varepsilon_0}\rho \qquad \phi^{in}(\rho) = -\frac{\rho_c}{4\varepsilon_0}\rho^2 + b$$

$$E_r^{out} = -\frac{c}{\rho} \qquad \phi^{out}(\rho) = c\ln\rho + d$$

Now we must determine the three constants *b, c* and *d*.
We have three conditions:
(1) Continuity of the electric field at ρ =R
(2) The potential at ρ =R
(3) Continuity of the potential at ρ =R

(1) $$E_r^{in}(R) = E_r^{out}(R) \;\Rightarrow\; +\frac{\rho_c}{2\varepsilon_0}R = -\frac{c}{R} \;\Rightarrow\; c = -\frac{\rho_c}{2\varepsilon_0}R^2 \qquad \Rightarrow \phi^{out}(\rho) = -\frac{\rho_c}{2\varepsilon_0}R^2\ln\rho + d$$

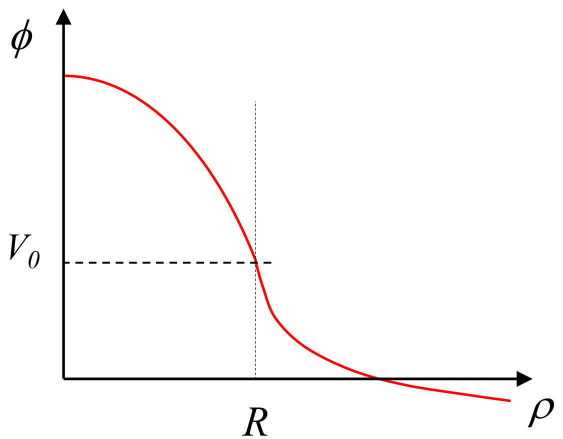
(2) $$\phi^{out}(R) = V_0 \;\Rightarrow\; -\frac{\rho_c}{2\varepsilon_0}R^2\ln R + d = V_0 \;\Rightarrow\; d = V_0 + \frac{\rho_c}{2\varepsilon_0}R^2\ln R$$

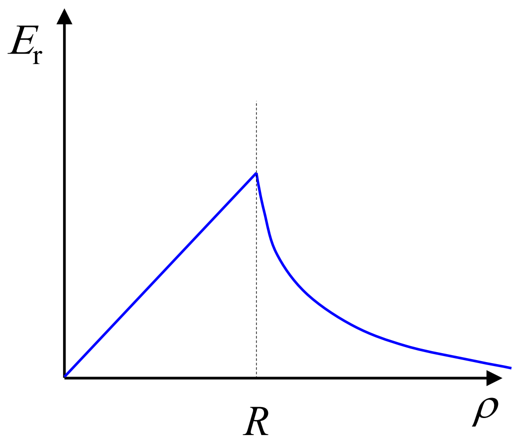
$$\Rightarrow \phi^{out}(\rho) = -\frac{\rho_c}{2\varepsilon_0}R^2\ln\rho + d = -\frac{\rho_c}{2\varepsilon_0}R^2\ln\rho + V_0 + \frac{\rho_c}{2\varepsilon_0}R^2\ln R = V_0 - \frac{\rho_c}{2\varepsilon_0}R^2\ln\left(\frac{\rho}{R}\right)$$

(3) $$\phi^{out}(R) = \phi^{in}(R) \;\Rightarrow\; V_0 - \frac{\rho_c}{2\varepsilon_0}R^2\underbrace{\ln\left(\frac{R}{R}\right)}_{=0} = -\frac{\rho_c}{4\varepsilon_0}R^2 + b \;\Rightarrow\; b = V_0 + \frac{\rho_c}{4\varepsilon_0}R^2$$

$$\phi^{in}(\rho) = V_0 + \frac{\rho_c}{4\varepsilon_0} R^2 \left(1 - \frac{\rho^2}{R^2}\right) \qquad E_r^{in}(\rho) = +\frac{\rho_c}{2\varepsilon_0}\rho$$

$$\phi^{out}(\rho) = V_0 - \frac{\rho_c}{2\varepsilon_0} R^2 \ln\left(\frac{\rho}{R}\right) \qquad E_r^{out} = \frac{\rho_c}{2\varepsilon_0}\frac{R^2}{\rho}$$





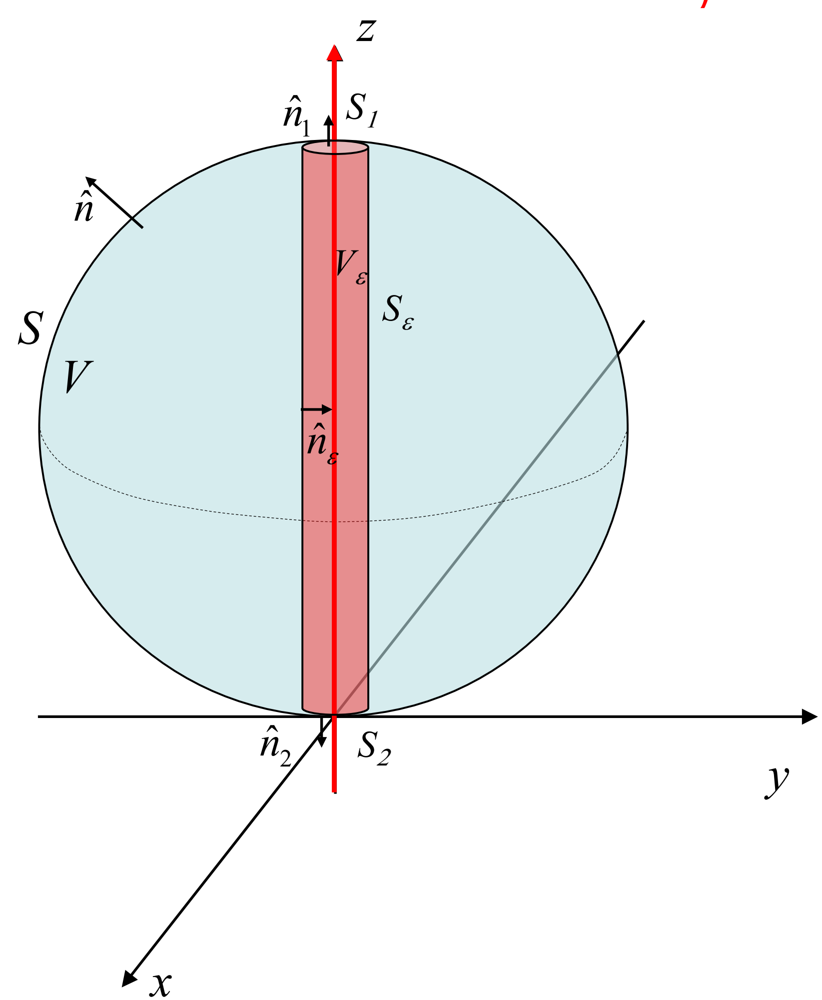
Note that: $\lim_{\rho\to\infty} \phi(\rho) = -\infty$

So, this solution is unphyiscal.
How is this possible? This is because infinitely long cylinders do not exist.
Nonetheless, the solution can be considered as a reasonable approximation for cylinders with $R \ll L$ (where L is the length of the cylinder).

PROBLEM 4

Use the Gauss theorem to calculate the flux of the vector field: $\bar{A}(\rho,\varphi,z) = z\dfrac{\rho^2-1}{\rho}\hat{e}_\rho$

on the surface S: $x^2+y^2+(z-2)^2=4$



SOLUTION

The field is singular at $\rho=0$ (the z-axis)
The Gauss theorem cannot be applied on S.

We divide V into two volumes:

$V=V_0+V_\varepsilon$

Thin cylinder with radius ε along the z-axis

The surface, boundary to V_0, is: $S+S_\varepsilon-S_1-S_2$

V_0 *does not contain the z-axis*
$\Rightarrow$ *we can apply Gauss*

$$\iint_S \bar{A}(\bar{r})\cdot d\bar{S} = \iint_{\substack{S+S_\varepsilon-S_\varepsilon+S_1\\-S_1+S_2-S_2}} \bar{A}(\bar{r})\cdot d\bar{S} =$$

$$= \iint_{S+S_\varepsilon-S_1-S_2} \bar{A}(\bar{r})\cdot d\bar{S} + \iint_{-S_\varepsilon+S_1+S_2} \bar{A}(\bar{r})\cdot d\bar{S} = \iiint_{V_0} div\bar{A}dV + \iint_{-S_\varepsilon+S_1+S_2} \bar{A}(\bar{r})\cdot d\bar{S}$$

$$=\iiint\limits_{V_0} div\overline{A}dV+\iint\limits_{-S_\varepsilon+S_1+S_2}\overline{A}(\overline{r})\cdot d\overline{S}=\underbrace{\iiint\limits_{V_0} div\overline{A}dV}_{I}\underbrace{-\iint\limits_{S_\varepsilon}\overline{A}(\overline{r})\cdot d\overline{S}}_{II}+\underbrace{\iint\limits_{S_1}\overline{A}(\overline{r})\cdot d\overline{S}}_{III}+\underbrace{\iint\limits_{S_2}\overline{A}(\overline{r})\cdot d\overline{S}}_{IV}$$

Integrals *III* and *IV* are zero:

$$\iint\limits_{S_1}\overline{A}(\overline{r})\cdot d\overline{S}=\iint\limits_{S_1} z\frac{\rho^2-1}{\rho}\underbrace{\hat{e}_\rho\cdot\hat{e}_z}_{=0}dS=\boxed{0}$$

$$\lim_{\varepsilon\to0} d\overline{S}=\hat{e}_z dS$$

$$\iint\limits_{S_2}\overline{A}(\overline{r})\cdot d\overline{S}=-\iint\limits_{S_2} z\frac{\rho^2-1}{\rho}\underbrace{\hat{e}_\rho\cdot\hat{e}_z}_{=0}dS=\boxed{0}$$

$$\lim_{\varepsilon\to0} d\overline{S}=-\hat{e}_z dS$$

Integrals *II* is:

$$d\overline{S}=-\hat{e}_\rho dS \qquad dS=\varepsilon d\varphi dz$$

because on S_ε: $\rho=\varepsilon$

$$-\iint\limits_{S_\varepsilon}\overline{A}(\overline{r})\cdot d\overline{S}=-\iint\limits_{S_\varepsilon} z\frac{\rho^2-1}{\rho}\hat{e}_\rho\cdot\left(-\hat{e}_\rho\right)dS=\iint\limits_{S_\varepsilon} z\frac{\rho^2-1}{\rho}dS=\iint\limits_{S_\varepsilon} z\frac{\varepsilon^2-1}{\varepsilon}\varepsilon d\varphi dz=$$

$$=\int_0^{2\pi}\int_{2-\sqrt{4-\varepsilon^2}}^{2+\sqrt{4-\varepsilon^2}} z\left(\varepsilon^2-1\right)d\varphi dz=2\pi\left(\varepsilon^2-1\right)\int_{2-\sqrt{4-\varepsilon^2}}^{2+\sqrt{4-\varepsilon^2}} zdz=8\pi\left(\varepsilon^2-1\right)\sqrt{4-\varepsilon^2}$$

But the cylinder is very *"thin"*, which means that we need to calculate the limit for : $\varepsilon \to 0$

$$\lim_{\varepsilon \to 0} 8\pi\left(\varepsilon^2 - 1\right)\sqrt{4-\varepsilon^2} = \boxed{-16\pi}$$

Integrals I is:

$$\iiint_{V_0} div\overline{A}dV = \iiint_{V_0} div\left(z\frac{\rho^2-1}{\rho}\hat{e}_\rho\right)dV = \iiint_{V_0} \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho z\frac{\rho^2-1}{\rho}\right)dV = \iiint_{V_0} 2zdV$$

The volume is a sphere with centre in the point (0,0,2)

Therefore, to use spherical coord. a coordinate transformation is necessary: z'=z-2

$$\iiint_{V_0} 2zdV = \iiint_{V_0} 2(z'+2)dV = \iiint_{V_0} 4dV + \iiint_{V_0} 2z'dV = 4\frac{4\pi 2^3}{3} + \iiint_{V_0} 2r\cos\theta r^2 \sin\theta drd\theta d\varphi$$

$$= \frac{128\pi}{3} + 2\left[\frac{r^4}{4}\right]_0^2 2\pi \underbrace{\left[-\frac{\cos^2\theta}{2}\right]_0^\pi}_{0} = \boxed{\frac{128\pi}{3}}$$

$$I + II + III + IV = \frac{128\pi}{3} - 16\pi + 0 + 0 = \frac{80\pi}{3}$$