VEKTORANALYS

HT 2021

CELTE / CENMI

ED1110

NABLAOPERATOR och NABLARÄKNING, INTEGRALSATSER, TENSORER och INDEXRÄKNING

ÖVNINGAR

Kapitel 11, 12, 14
Kapitel 15

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PROBLEM 1

Use “nablaräkning” to verify: $rot\left(\phi\overline{A}\right) = grad\,\phi\times\overline{A} + \phi\, rot\overline{A}$ ID3

SOLUTION

Add the dots

$$rot\left(\phi\overline{A}\right) = \nabla\times\left(\phi\overline{A}\right) = \nabla\times\left(\underset{\cdot}{\phi}\overline{A}\right) + \nabla\times\left(\phi\underset{\cdot}{\overline{A}}\right) =$$

Now nabla can be considered as a vector: $\overline{n}\times\left(c\overline{a}\right) + \overline{n}\times\left(c\overline{a}\right) = \left(\overline{n}c\right)\times\overline{a} + c\left(\overline{n}\times\overline{a}\right)$ *(because c is a scalar)*

$$= \left(\nabla\underset{\cdot}{\phi}\right)\times\overline{A} + \phi\left(\nabla\times\underset{\cdot}{\overline{A}}\right) =$$

$$= \left(\nabla\phi\right)\times\overline{A} + \phi\left(\nabla\times\overline{A}\right)$$

PROBLEM 2

Use “nablaräkning” to verify: $div\, rot\overline{A} = 0$ ID8

SOLUTION

Now nabla can be considered as a vector.

$$div\, rot\overline{A} = \nabla\cdot\left(\nabla\times\underset{\cdot}{\overline{A}}\right) =$$

Because: $\overline{n}\cdot\left(\overline{n}\times\overline{a}\right) = \overline{a}\cdot\left(\underbrace{\overline{n}\times\overline{n}}_{=0}\right)$

$$= \underset{\cdot}{\overline{A}}\cdot\left(\nabla\times\nabla\right) = \left(\nabla\times\nabla\right)\cdot\underset{\cdot}{\overline{A}} = 0$$

PROBLEM 3

Use “nablaräkning” to verify: $\left(\bar{A}\times\nabla\right)\times\bar{A}=\frac{1}{2}\nabla A^2-\bar{A}\left(\nabla\cdot\bar{A}\right)$

SOLUTION

$$\left(\bar{A}\times\nabla\right)\times\bar{A}=\left(\bar{A}\times\nabla\right)\times\underset{\cdot}{\bar{A}}=$$

$$(\bar{a}\times\bar{n})\times\bar{b}=(\bar{a}\cdot\bar{b})\bar{n}-(\bar{n}\cdot\bar{b})\bar{a}$$

$$=\nabla\left(\bar{A}\cdot\underset{\cdot}{\bar{A}}\right)-\bar{A}\left(\nabla\cdot\underset{\cdot}{\bar{A}}\right)=$$

$$\nabla A^2=\nabla\left(\bar{A}\cdot\bar{A}\right)=\nabla\left(\underset{\cdot}{\bar{A}}\cdot\bar{A}\right)+\nabla\left(\bar{A}\cdot\underset{\cdot}{\bar{A}}\right)=2\nabla\left(\bar{A}\cdot\underset{\cdot}{\bar{A}}\right)$$

$$\Rightarrow\quad\nabla\left(\bar{A}\cdot\underset{\cdot}{\bar{A}}\right)=\frac{1}{2}\nabla A^2$$

$$=\frac{1}{2}\nabla A^2-\bar{A}\left(\nabla\cdot\bar{A}\right)$$

PROBLEM 4

The divergence of the magnetic field is zero: $\nabla\cdot\bar{B}=0$

Use “nablaräkning” to prove: $\bar{B}\times\left(\nabla\times\bar{B}\right)-\left(\bar{B}\times\nabla\right)\times\bar{B}=-\left(\bar{B}\cdot\nabla\right)\bar{B}$

SOLUTION

$$\bar{B}\times\left(\nabla\times\bar{B}\right)-\left(\bar{B}\times\nabla\right)\times\bar{B}=\bar{B}\times\left(\nabla\times\underset{\cdot}{\bar{B}}\right)-\left(\bar{B}\times\nabla\right)\times\underset{\cdot}{\bar{B}}=\bar{B}\times\left(\nabla\times\underset{\cdot}{\bar{B}}\right)+\underset{\cdot}{\bar{B}}\times\left(\bar{B}\times\nabla\right)=$$

$$=\nabla\left(\bar{B}\cdot\underset{\cdot}{\bar{B}}\right)-\underset{\cdot}{\bar{B}}\left(\bar{B}\cdot\nabla\right)-\nabla\left(\underset{\cdot}{\bar{B}}\cdot\bar{B}\right)+\bar{B}\left(\underset{\cdot}{\bar{B}}\cdot\nabla\right)=$$

using $\bar{a}\times\left(\bar{b}\times\bar{c}\right)=\bar{b}\left(\bar{a}\cdot\bar{c}\right)-\bar{c}\left(\bar{a}\cdot\bar{b}\right)$

$$=\nabla\left(\underset{\cdot}{\bar{B}}\cdot\bar{B}\right)-\left(\bar{B}\cdot\nabla\right)\underset{\cdot}{\bar{B}}-\nabla\left(\underset{\cdot}{\bar{B}}\cdot\bar{B}\right)+\bar{B}\left(\nabla\cdot\underset{\cdot}{\bar{B}}\right)=$$

$$=-\left(\bar{B}\cdot\nabla\right)\underset{\cdot}{\bar{B}}+\bar{B}\left(\nabla\cdot\underset{\cdot}{\bar{B}}\right)=\bar{B}\underbrace{\left(\nabla\cdot\bar{B}\right)}_{=0}-\left(\bar{B}\cdot\nabla\right)\bar{B}=-\left(\bar{B}\cdot\nabla\right)\bar{B}$$

PROBLEM 5

Calculate $\varepsilon_{ijk}\varepsilon_{ljk}$

SOLUTION

We know that $\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

Therefore:

the same expression with m=j

We re-arrange the suffixes to have an expression similar to

$$\varepsilon_{ijk}\varepsilon_{ljk} = \varepsilon_{ijk}\varepsilon_{klj} = \delta_{il}\delta_{jj} - \delta_{ij}\delta_{jl} = \delta_{il}3 - \delta_{il} = 2\delta_{il}$$

even permutations does NOT change the sign

Remember that $\delta_{ii} = 3$

Remember that $\delta_{km}p_m = p_k$

PROBLEM 6

Prove $\overline{a}\times\left(\overline{b}\times\overline{c}\right)=\left(\overline{a}\cdot\overline{c}\right)\overline{b}-\left(\overline{a}\cdot\overline{b}\right)\overline{c}$

using the suffix notation.

SOLUTION

We know that the *i-component* of the cross product can be written as: $\left(\overline{a}\times\overline{b}\right)_i=\varepsilon_{ijk}a_jb_k$

Therefore:

$$\left(\overline{b}\times\overline{c}\right)_k=\varepsilon_{klm}b_lc_m$$

$$\left(\overline{a}\times\left(\overline{b}\times\overline{c}\right)\right)_i=\varepsilon_{ijk}a_j\left(\overline{b}\times\overline{c}\right)_k=$$

$$\varepsilon_{ijk}\varepsilon_{klm}=\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}$$

$$=\varepsilon_{ijk}a_j\varepsilon_{klm}b_lc_m=$$

$$\delta_{ij}d_{kj}=d_{ki}$$

$$=\varepsilon_{ijk}\varepsilon_{klm}a_jb_lc_m=$$

$$=\left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right)a_jb_lc_m=\delta_{il}\delta_{jm}a_jb_lc_m-\delta_{im}\delta_{jl}a_jb_lc_m=$$

$$=a_mb_ic_m-a_lb_lc_i=\left(\overline{a}\cdot\overline{c}\right)b_i-\left(\overline{a}\cdot\overline{b}\right)c_i$$

PROBLEM 7

Use “indexräkning” to verify: $div\left(\overline{A}\times\overline{B}\right)=\left(rot\overline{A}\right)\cdot\overline{B}-\left(rot\overline{B}\right)\cdot\overline{A}$ **ID4**

SOLUTION

$$div\left(\overline{A}\times\overline{B}\right)=\left(\overline{A}\times\overline{B}\right)_{i,i}=\left(\varepsilon_{ijk}A_jB_k\right)_{,i}=\varepsilon_{ijk}\left(A_jB_k\right)_{,i}=\varepsilon_{ijk}\left(A_{j,i}B_k+A_jB_{k.i}\right)=$$

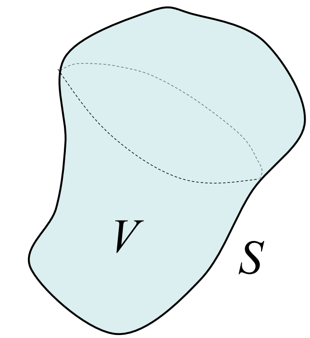
$$=\varepsilon_{ijk}A_{j,i}B_k+\varepsilon_{ijk}A_jB_{k,i}=\varepsilon_{kij}A_{j,i}B_k-\varepsilon_{jik}B_{k,i}A_j=\left(rot\overline{A}\right)\cdot\overline{B}-\left(rot\overline{B}\right)\cdot\overline{A}$$

remember that: $\left(\nabla\times\overline{A}\right)_i=\varepsilon_{ijk}A_{k,j}$

PROBLEM 8

Show that: $$\iiint_V \overline{r} \times rot\overline{A}\,dV = 2\iiint_V \overline{A}\,dV$$

if on the boundary surface S of V the vector field is $\overline{A} = 0$



SOLUTION

Let's consider only the *i-th* component of the left hand side:

$$\hat{e}_i \cdot \iiint_V \overline{r} \times rot\overline{A}\; dV = \iiint_V \hat{e}_i \cdot \left(\overline{r} \times rot\overline{A}\right) dV = \iiint_V \left(rot\overline{A}\right) \cdot \left(\hat{e}_i \times \overline{r}\right) dV =$$

$$\overline{a} \cdot \left(\overline{b} \times \overline{c}\right) = \overline{c} \cdot \left(\overline{a} \times \overline{b}\right)$$

and we substitute

To continue, we must remember that: $\nabla \cdot \left(\overline{a} \times \overline{b}\right) = \overline{b} \cdot \left(\nabla \times \overline{a}\right) - \overline{a} \cdot \left(\nabla \times \overline{b}\right)$ (**ID4**)

therefore, $\nabla \cdot \left(\overline{A} \times \left(\hat{e}_i \times \overline{r}\right)\right) = \left(\hat{e}_i \times \overline{r}\right) \cdot \left(\nabla \times \overline{A}\right) - \overline{A} \cdot \nabla \times \left(\hat{e}_i \times \overline{r}\right)$

re-arranging the terms: $\left(rot\overline{A}\right) \cdot \left(\hat{e}_i \times \overline{r}\right) = div\left(\overline{A} \times \left(\hat{e}_i \times \overline{r}\right)\right) + \overline{A} \cdot rot\left(\hat{e}_i \times \overline{r}\right)$

$$= \iiint_V \left[div\left(\overline{A}\times\left(\hat{e}_i\times\overline{r}\right)\right) + \overline{A}\cdot rot\left(\hat{e}_i\times\overline{r}\right) \right] dV =$$

$$= \iiint_V div\left(\overline{A}\times\left(\hat{e}_i\times\overline{r}\right)\right) dV + \iiint_V \overline{A}\cdot rot\left(\hat{e}_i\times\overline{r}\right) dV =$$

Generalized Gauss theorem

ID5

$$rot\left(\hat{e}_i\times\overline{r}\right) = \left(\overline{r}\cdot\nabla\right)\hat{e}_i - \left(\hat{e}_i\cdot\nabla\right)\overline{r} + \hat{e}_i\left(\nabla\cdot\overline{r}\right) - \overline{r}\left(\nabla\cdot\hat{e}_i\right)$$

$$= \quad 0 \quad - \frac{\partial\overline{r}}{\partial x_i} \quad + \quad 3\hat{e}_i \quad - \quad 0 \quad = 2\hat{e}_i$$

$$= \underbrace{\iint_S \left(\overline{A}\times\left(\hat{e}_i\times\overline{r}\right)\right)\cdot d\overline{S}}_{=0} + \iiint_V \overline{A}\cdot 2\hat{e}_i\, dV = 2\iiint_V A_i\, dV$$

Because on S, $\overline{A}=0$

So, we have: $$\hat{e}_i\cdot\iiint_V \overline{r}\times rot\overline{A}\, dV = 2\iiint_V A_i\, dV$$