VEKTORANALYS

HT 2021

CELTE / CENMI

ED1110

GAUSS' THEOREM and STOKES' THEOREM: ÖVNINGAR

Kursvecka 3

Kapitel 8-9 (*Vektoranalys,* 1:e uppl, Frassinetti/Scheffel)

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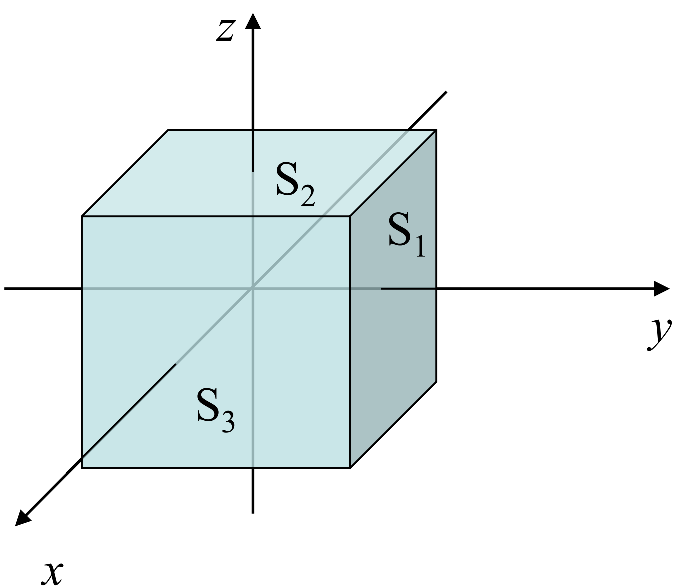
PROBLEM 1

Calculate: $\iint_S \overline{A} \cdot d\overline{S}$ where the vector field is: $\overline{A} = (x, y, z)$

and S is a cube (length 2 each side) centred in the origin.
The normal to the surface points out from the cube

(a) In a direct way (using the parameterization of the surface).
(b) Using Gauss' theorem

SOLUTION



(a) $$\iint_S \overline{A} \cdot d\overline{S} = \sum_i \iint_{S_i} \overline{A} \cdot d\overline{S}$$

Let's start with S_1

1- parameterization of S_1:

$$\left.\begin{array}{l} y = 1 \\ |x| < 1 \\ |z| < 1 \end{array}\right\} \Rightarrow \underset{\substack{u:-1\to+1 \\ v:-1\to+1}}{\overline{r}(u,v)} = (u, 1, v)$$

2- Integral calculation:

$$\int_{S_1} \overline{A}\cdot d\overline{S} = \int_u \int_v \overline{A}\left(\overline{r}(u,v)\right)\cdot\left(\frac{\partial\overline{r}}{\partial u}\times\frac{\partial\overline{r}}{\partial v}\right)dudv$$

$$\left.\begin{array}{l}\dfrac{\partial\overline{r}}{\partial u} = (1,0,0)\\[2ex] \dfrac{\partial\overline{r}}{\partial v} = (0,0,1)\end{array}\right\} \Rightarrow \left(\frac{\partial\overline{r}}{\partial u}\times\frac{\partial\overline{r}}{\partial v}\right) = (0,-1,0)$$

The normal is pointing inside the volume → remember to change the sign of the flow in the final answer.

$$\int_{-S_1} \overline{A}\cdot d\overline{S} = \int_{-1}^{1}\int_{-1}^{1} (u,1,v)\cdot(0,-1,0)dudv = \int_{-1}^{1}\int_{-1}^{1} dudv = -4$$

Due to the symmetry of the problem we have: $\int_{-S_i} \overline{A}\cdot d\overline{S} = -4$

$$\Rightarrow \iint_{-S} \overline{A}\cdot d\overline{S} = \sum_i \iint_{-S_i} \overline{A}\cdot d\overline{S} = 6\cdot(-4) = -24$$

We need to change the sign of the flow: $\Rightarrow \iint_S \overline{A}\cdot d\overline{S} = 24$

(b) S is a closed surface and $\overline{A}$ is continuous and differentiable $\Rightarrow$ we can apply Gauss' theorem

$$\left.\begin{array}{l}\displaystyle\iint_S \overline{A}\cdot d\overline{S} = \iiint_V div\overline{A}dV\\[3ex] div\overline{A} = \dfrac{\partial A_x}{\partial x}+\dfrac{\partial A_y}{\partial y}+\dfrac{\partial A_z}{\partial z} = 1+1+1 = 3\end{array}\right\} \Rightarrow \iint_S \overline{A}\cdot d\overline{S} = \iiint_V 3dV = 3V = 3\cdot 2^3 = 24$$

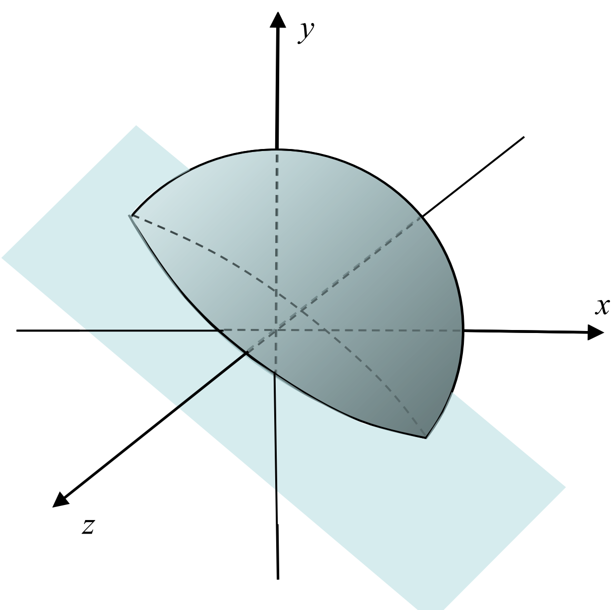
PROBLEM 2

Calculate $\iint_S \overline{A} \cdot d\overline{S}$ using Gauss' theorem

where the vector field is: $\overline{A} = (x^3, y^3, z^3)$

with S an open surface defined as the half sphere: $\begin{cases} x^2 + y^2 + z^2 = R^2 \\ x + y > 0 \end{cases}$

SOLUTION



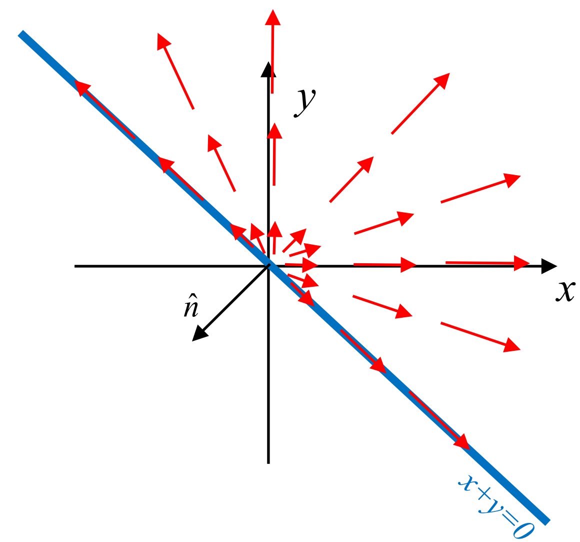
But S is NOT a closed surface!
So we can consider the surface
$S_{tot} = S + S_{plane}$

$$\oiint_{S_{tot}} \overline{A} \cdot d\overline{S} = \iiint_V div\overline{A} dV$$

$$\iint_S \overline{A} \cdot d\overline{S} = \oiint_{S_{tot}} \overline{A} \cdot d\overline{S} - \iint_{S_{plane}} \overline{A} \cdot d\overline{S}$$

$$\iint_S \overline{A}\cdot d\overline{S} = \iiint_V div\overline{A}dV - \iint_{S_{plane}} \overline{A}\cdot d\overline{S}$$

So we have transformed a surface integral
into a volume integral minus another surface integral
What is the advantage?
They can be calculated much easier!!



Let's consider the second integral.

$$S_{plane} \quad \begin{cases} x^2+y^2+z^2 \le R^2 \\ x+y=0 \end{cases}$$

On S_{plane} $\quad x=-y$

$$\overline{A} = (x^3, y^3, z^3) \quad \Rightarrow \quad \overline{A} = (x^3, -x^3, z^3)$$

On S_{plane} the vector is perpendicular to $\hat{n} \quad \Rightarrow \iint_{S_{plane}} \overline{A}\cdot d\overline{S} = 0$

Let's consider the first integral.

Spherical coordinates

$$\iiint_V div\overline{A}dV \quad with \quad div\overline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 3x^2+3y^2+3z^2 = 3r^2$$

since $div\overline{A} = 3r^2 \overset{\text{due to symmetry}}{\Rightarrow} \iiint\limits_V div\overline{A}dV = \frac{1}{2}\iiint\limits_{V_{sphere}} div\overline{A}dV$

$$\iiint\limits_{V_{sphere}} div\overline{A}dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} 3r^2\ r^2 \sin\theta dr d\theta d\varphi = 3\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \int_0^{R} r^4\ dr = \frac{12\pi R^5}{5}$$

$$\Rightarrow \quad \iiint\limits_V div\overline{A}dV = \frac{1}{2}\iiint\limits_{V_{sphere}} div\overline{A}dV = \frac{6\pi R^5}{5}$$

$$\iint\limits_S \overline{A}\cdot d\overline{S} = \iiint\limits_V div\overline{A}dV - \iint\limits_{S_{plane}} \overline{A}\cdot d\overline{S} = \frac{6\pi R^5}{5}$$

PROBLEM 3 (example 9.2 in the book)

Med hjälp av Stokes sats, beräkna linjeintegralen av vektorfältet

$$\overline{A}(\overline{r}) = x\hat{e}_y$$

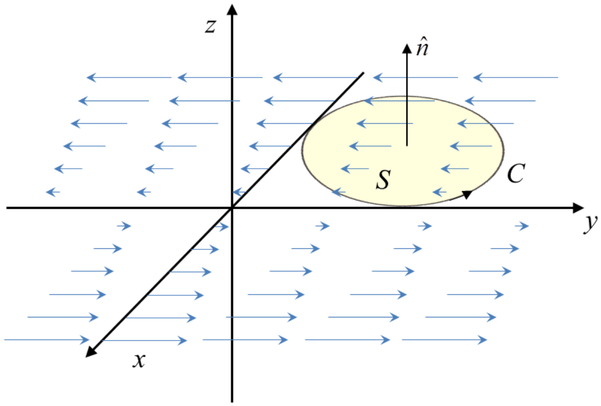
längs cirkeln C med radie 1, centrerad i punkten $(-1,1,0)$. Cirkeln ligger i xy-planet och har en kontur som är orienterad moturs, sett från positiva z-axeln.

SOLUTION

$$\oint_C \overline{A}\cdot d\overline{r} = \iint_S rot\overline{A}\cdot d\overline{S}$$

$$rot\overline{A} = rot\left(x\hat{e}_y\right) = \left(-\frac{\partial x}{\partial z}\right)\hat{e}_x + \left(\frac{\partial x}{\partial x}\right)\hat{e}_z = \hat{e}_z$$

For S we can choose any surface that has the boundary on C. The simplest choice is the surface corresponding to the area that lies in xy-plane inside C.
According to the right hand rule, the normal will be in the $\hat{e}_z$ direction



$$\oint_C \overline{A}\cdot d\overline{r} = \iint_S rot\overline{A}\cdot d\overline{S} = \iint_S \hat{e}_z\cdot d\overline{S} = \iint_S \hat{e}_z\cdot\hat{e}_z dS = \iint_S dS = \pi$$

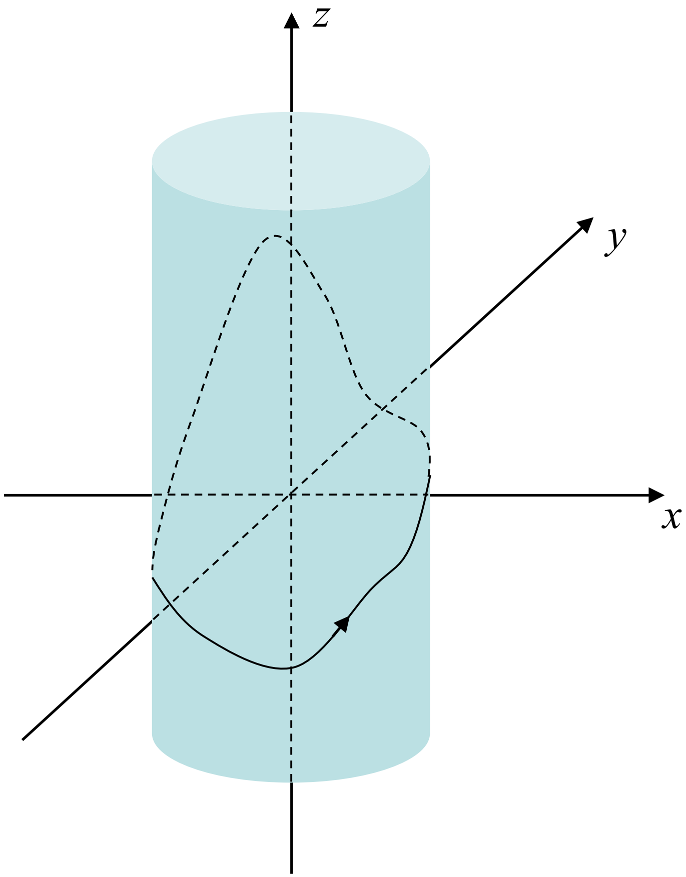
PROBLEM 4

Calculate the line integral of the vector field: $\overline{A} = \left(y + 2x,\, x^2 + z,\, y\right)$

along the closed curve: $\begin{cases} \overline{r}(u) = (\cos u, \sin u, f(u)) \quad with \quad f(u) = f(u + 2\pi) \\ u : \quad 0 \to 2\pi \end{cases}$

(a) directly
(b) using Stokes' theorem

SOLUTION



The curve is on the cylinder defined by *(cos u, sin u, z)*

On the cylinder the curve Is defined by $z=f(u)$

SOLUTION (A)

We will calculate $\displaystyle \int_L \overline{A}\cdot d\overline{r} = \int_a^b \overline{A}(\overline{r}(u))\cdot \frac{d\overline{r}}{du}\,du$

$$\frac{d\overline{r}}{du} = \left(-\sin u, \cos u, \frac{df}{du}\right)$$

$$\overline{A}(\overline{r}(u)) = \left(\sin u + 2\cos u, \cos^2 u + f(u), \sin u\right)$$

$$\Rightarrow \int_L \overline{A}\cdot d\overline{r} = \int_0^{2\pi} \left(\sin u + 2\cos u, \cos^2 u + f(u), \sin u\right)\cdot\left(-\sin u, \cos u, \frac{df}{du}\right) du =$$

$$-\int_0^{2\pi} \sin^2 u\,du - 2\int_0^{2\pi} \sin u\cos u\,du + \int_0^{2\pi} \cos^3 u\,du + \int_0^{2\pi}\left(f(u)\cos u + \frac{df}{du}\sin u\right) du =$$

$$-\left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_0^{2\pi} - \left[\sin^2 u\right]_0^{2\pi} + \left[\sin u - \frac{1}{3}\sin^3 u\right]_0^{2\pi} + \left[f(u)\sin u\right]_0^{2\pi} = -\pi$$

$\overline{S}_2 = \hat{n}_2 S_2$

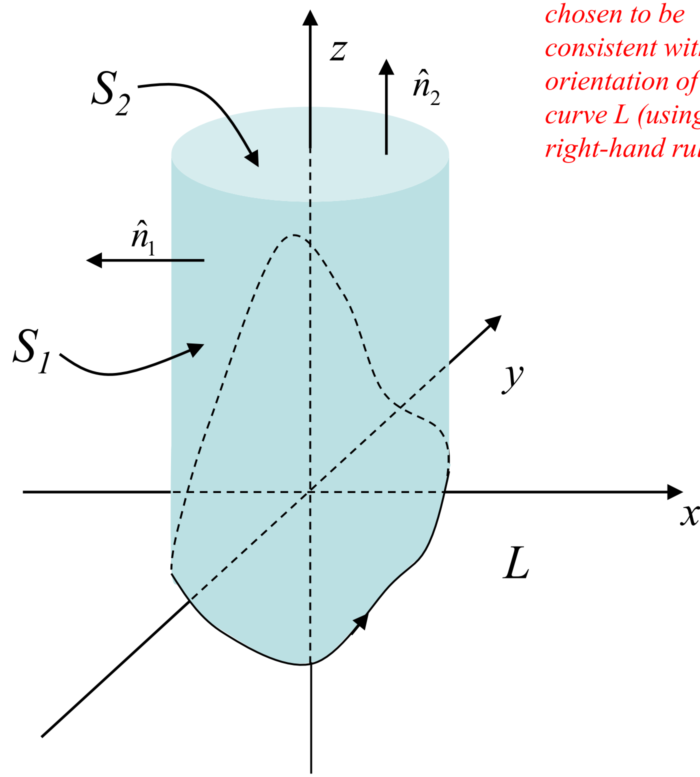
$\overline{S}_1 = \hat{n}_1 S_1$

Remember that the orientation of the normal must be chosen to be consistent with the orientation of the curve L (using the right-hand rule)

SOLUTION (B)

$$\int_L \overline{A}\cdot d\overline{r} = \iint_S rot\,\overline{A}\cdot d\,\overline{S}$$

$$S=S_1+S_2$$



$$rot\,\overline{A} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y+2x & x^2+z & y \end{vmatrix} = \left(1-1, 0-0, 2x-1\right) = (0,0,2x-1)$$

$$rot\,\overline{A} \text{ is in the z-direction} \Rightarrow \iint_{S_1} rot\,\overline{A}\cdot d\,\overline{S} = 0$$

$$\int_L \overline{A}\cdot d\overline{r} = \iint_S rot\,\overline{A}\cdot d\,\overline{S} = \iint_{S_2} rot\,\overline{A}\cdot d\,\overline{S}_2 = \iint_{S_2} (0,0,2x-1)\cdot \hat{e}_z dxdy = \iint_{S_2} (2x-1)dxdy$$

$\hat{e}_z$ is the direction of $\hat{n}_2$

cylindrical coord.

$$= \int_0^{2\pi}\int_0^1 (2\rho\cos\varphi - 1)\rho d\rho d\varphi = 2\int_0^{2\pi}\cos\varphi d\varphi \int_0^1 \rho^2 d\rho - \int_0^{2\pi} d\varphi \int_0^1 \rho d\rho = -2\pi\left[\frac{\rho^2}{2}\right]_0^1 = -\pi$$