VEKTORANALYS

HT 2021

CELTE / CENMI

ED1110

LINE INTEGRALS and FLUX INTEGRALS

Kursvecka 2

Kapitel 6-7 (*Vektoranalys*, 1:e uppl, Frassinetti/Scheffel)

version: 4-sept-2021

This week

- **Line integrals**
  - Most common line integral: meaning and technique to calculate it
  - Theorems on line integrals
  - Circulation, conservative fields and potential
  - Other types of line integrals
  - The Biot-Savart law (a practical application very useful for TET)

- **Flux integrals**
  - Meaning and definition
  - The technique to calculate a flux integral
  - The flux of the electric field through a sphere: a simple example very useful for TET

If we have time (Monday afternoon):

- **Solving 1-2 problems related to the Biot-Savart law:**

$$\overline{B}(\overline{r}) = \frac{\mu_0 I}{4\pi} \int_L \frac{d\overline{r}\,' \times (\overline{r} - \overline{r}\,')}{|\overline{r} - \overline{r}\,'|^3}$$

Connections with previous and next topics

Line integrals

- Vector fields (week 1)
- Curve parameterization and tangent vector (övningar in week 1)
- Conservative fields. (see also week 3)
- Stokes theorem (week 3)

It has important applications in physics and engineering: for example to calculate the work, to calculate the magnetic field produced by a wire (Biot-Savart law), the Ampere's law, the Faraday's law…

Flux integrals

- Surface parameterization and normal vector (övningar in week 1)
- Gauss' theorem (week 3)
- Special vector fields (week 6)

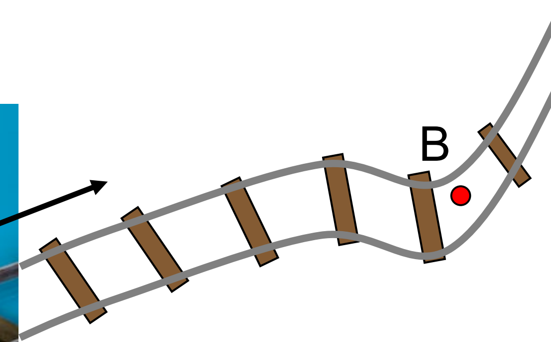
It has applications like the Guass' law for the electric field or to calculate the flux of any vector field on a surface

You are not allowed to use the words above to motivate and give examples in problem 1 of the home assignment

TARGET PROBLEM

A person is pushing a mine cart along a path L on a hill.

Calculate the "work" done to move the cart from A to B.
For simplicity, assume there is any friction.



PROBLEM

We have to push the cart along the path L defined as:

$$L: \begin{cases} \overline{r} = (\cos u, \sin u, u) \\ u : 0 \rightarrow 4\pi \end{cases}$$

(in a Cartesian coordinate system)

The weight (due to gravity force) points towards the negative direction of the z-axis. Moreover, we have to face a strong wind (whose intensity and direction depends on the position). Assume that the total force that we have to use is:

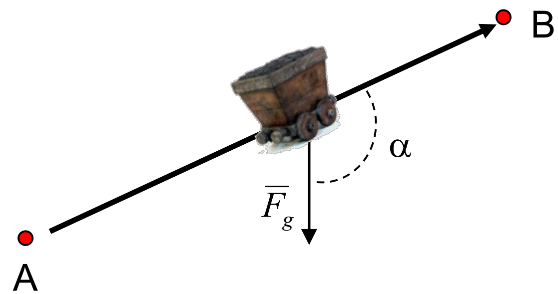
$$\overline{F} = (-yz, xz, -1)$$

(in a Cartesian coordinate system)

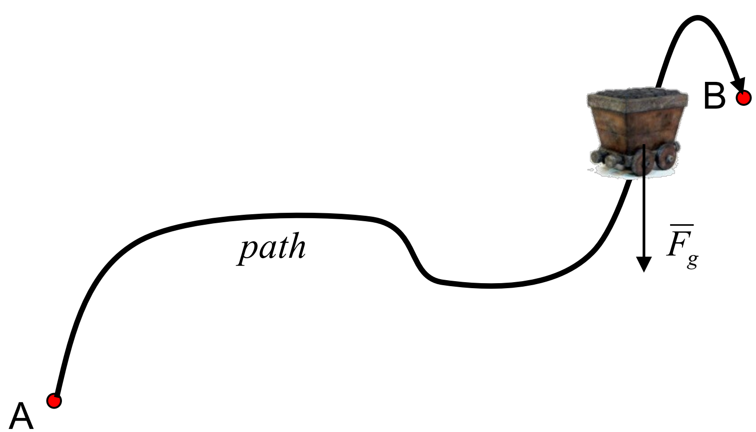
Calculate the work that we have to do to move the cart from the beginning to the end of the pat.

We will arrive to the final answer in two steps:

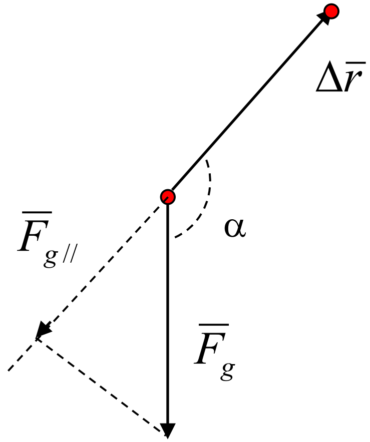
1- The slope is constant



2- the slope is not constant



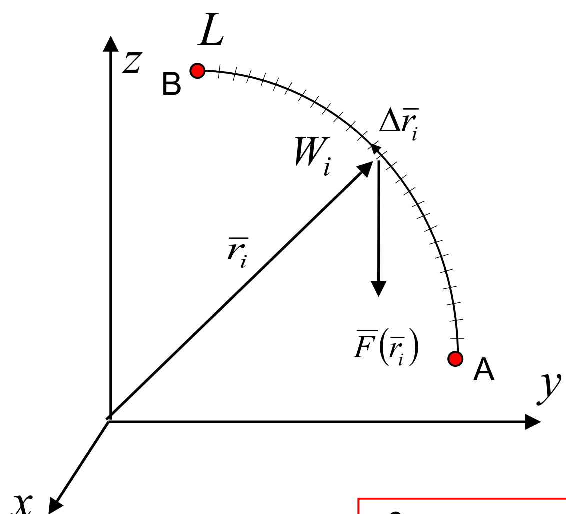
Step 1:



$$W = \left|\overline{F}_{g//}\right|\left|\Delta\overline{r}\right| = \left|\overline{F}_g\right|\left|\Delta\overline{r}\right|\cos\alpha = \overline{F}\cdot\Delta\overline{r}$$

(to be precise, we should change sign to calculate the work done by the person)

Step 2:



$$\left.\begin{array}{l} W = \sum_i W_i \\ W_i \approx \overline{F}(\overline{r}_i)\cdot\Delta\overline{r}_i \end{array}\right\} \Rightarrow W \approx \sum_i \overline{F}(\overline{r}_i)\cdot\Delta\overline{r}_i$$

For *"very small"* segments:

$$W = \lim_{\Delta\overline{r}_i\to 0}\sum_i \overline{F}(\overline{r}_i)\cdot\Delta\overline{r}_i \equiv \int_L \overline{F}(\overline{r})\cdot d\overline{r}$$

$\boxed{\int_L \overline{F}(\overline{r})\cdot d\overline{r}}$ is the line integral of $\overline{F}$ along the path L

We need to:
- introduce a **VECTOR FIELD**, $\overline{F}(\overline{r})$
- Define the **infinitesimal displacement** $d\overline{r}$ along the path L

In the slides we will show how to calculate a simple line integral.
We will calculate the following integral, step by step

$$\int_L \overline{F} \cdot d\overline{r}$$

with

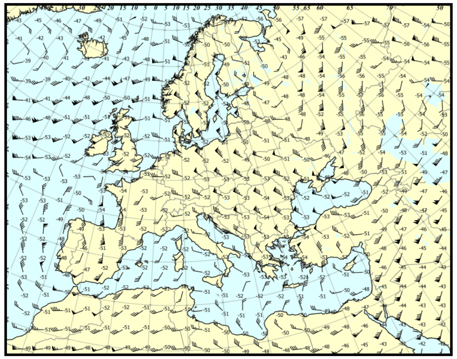
$$\overline{F} = x\hat{e}_x$$

$$L : \begin{cases} y = x^2 \\ from\ A:(0,0,0)\ to\ B:(1,1,0) \end{cases}$$

VECTOR FIELD

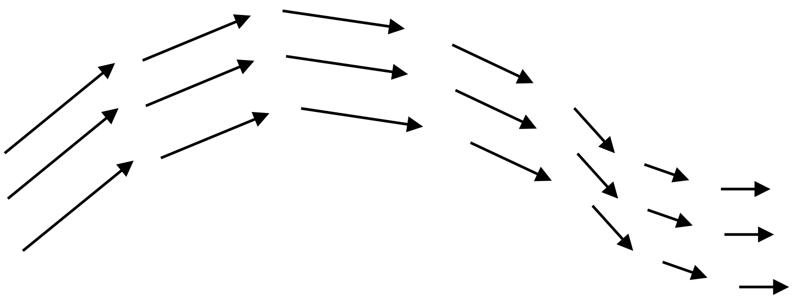
A **vector field** associates a vector $\bar{A}$(x,y,z) to each point (x,y,z) of the space.

Examples: - velocity distribution in a fluid
- magnetic field around a magnet
- electric field around an electric charge



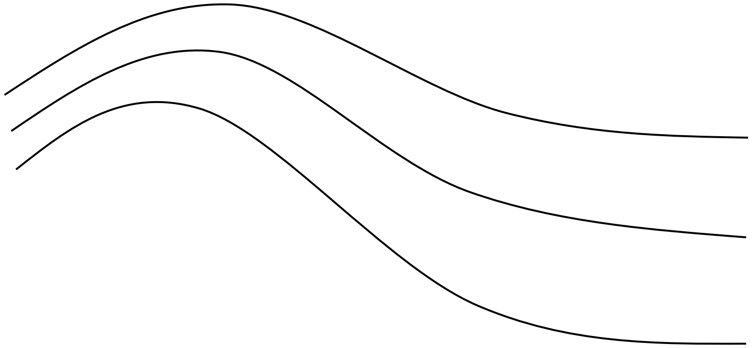
Two typical ways to represent a vector field:

1- Arrow field



- The arrow length is proportional to the field amplitude
- The arrow direction shows the field direction

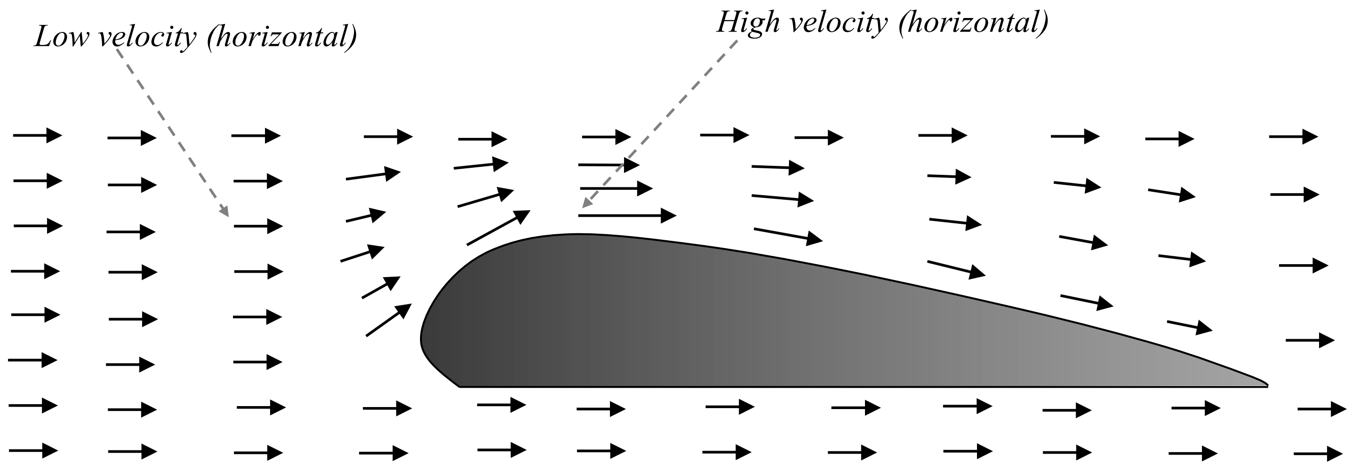
2- Line field

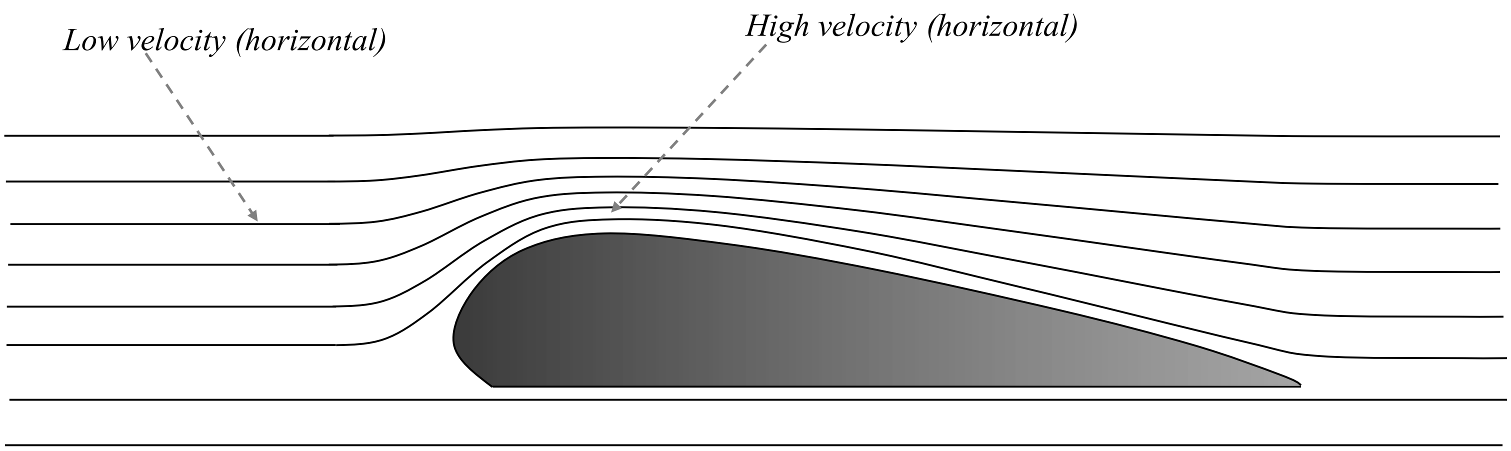


- The tangent to the curves is parallel to the vector field in all points.
- The density of the lines is proportional to the strength of the field.

VECTOR FIELD

The airplane wing example (velocity field of air around a wing)



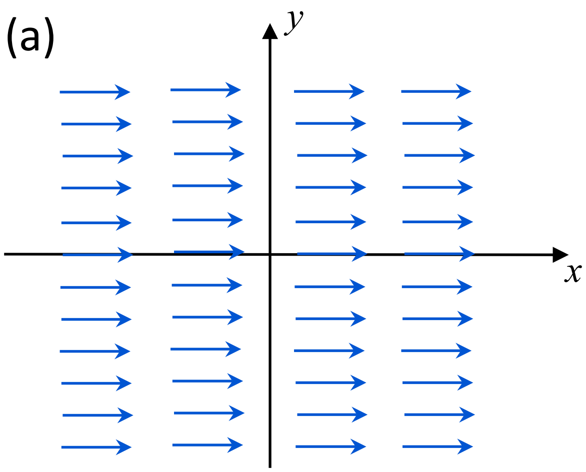


EXAMPLE

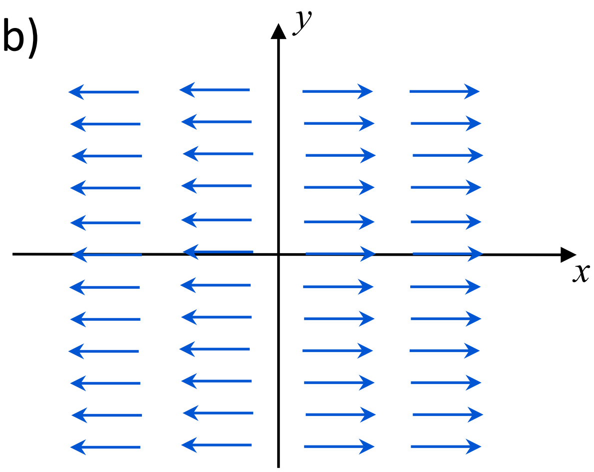
Consider the following vector field: $\bar{F}(\bar{r}) = x\hat{e}_x$

Which of the following figures can represent $\bar{F}$ in the z=0 plane?

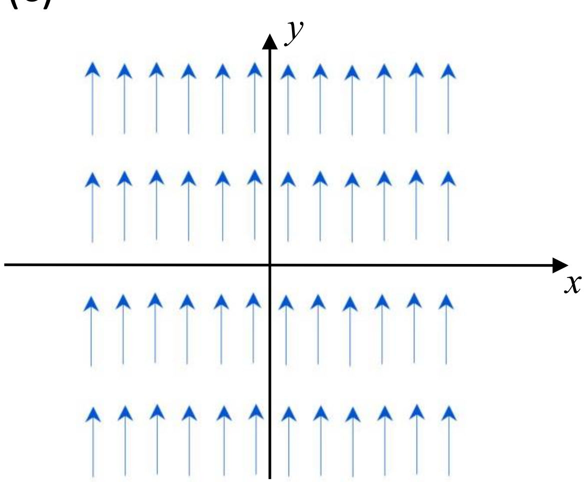
(a)



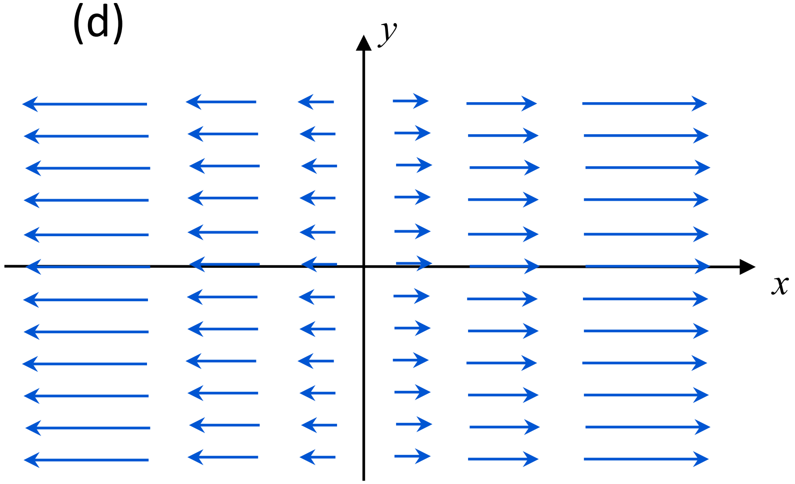
(b)



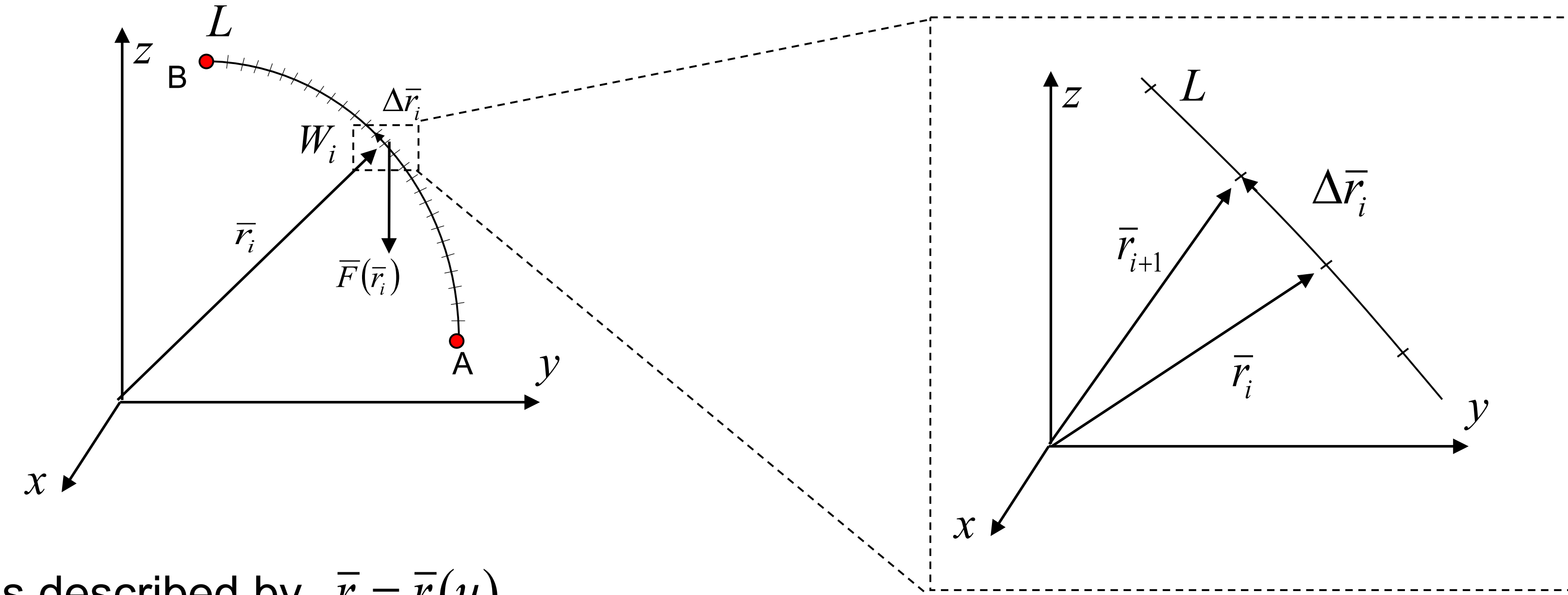
(c)



(d)



$d\bar{r}$ and THE LINE INTEGRAL



L is described by $\bar{r} = \bar{r}(u)$

$$\left.\begin{aligned} \bar{r}_i &= \bar{r}(u_i) \\ \bar{r}_{i+1} &= \bar{r}(u_{i+1}) \\ u_{i+1} &= u_i + \Delta u \end{aligned}\right\} \Rightarrow d\bar{r} = \lim_{\Delta u \to 0} \Delta\bar{r}_i = \lim_{\Delta u \to 0}\left[\bar{r}(u_{i+1}) - \bar{r}(u_i)\right] = \lim_{\Delta u \to 0} \frac{\left[\bar{r}(u_i + \Delta u) - \bar{r}(u_i)\right]}{\Delta u}\Delta u = \frac{d\bar{r}}{du}du$$

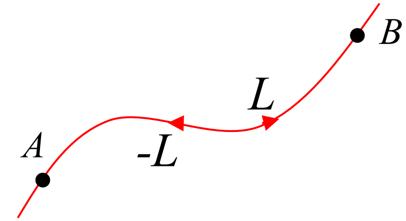
So, the line integral can be calculated as:

$$\boxed{\int_L \bar{F}(\bar{r}) \cdot d\bar{r} = \int_a^b \bar{F}(\bar{r}(u)) \cdot \frac{d\bar{r}}{du} du}$$

EXERCISE: Calculate $\int_L \bar{F}(\bar{r}) \cdot d\bar{r}$ with $\bar{F}(\bar{r}) = x\hat{e}_x$ on the curve $L: \begin{cases} y = x^2 \\ from\ A:(0,0,0)\ to\ B:(1,1,0) \end{cases}$

LINE INTEGRAL (some useful properties)

DEFINITION: The curve $-L$ has the same points as the curve L, but opposite direction



THEOREM 1

(6.1 in the textbook)

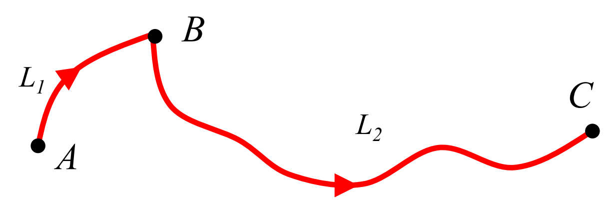
$$\boxed{\int_{-L} \overline{F}(\overline{r})\cdot d\overline{r} = -\int_{L} \overline{F}(\overline{r})\cdot d\overline{r}}$$

Proof:

$$\int_{L} \overline{F}(\overline{r})\cdot d\overline{r} = \int_a^b \overline{F}(\overline{r}(u))\cdot \frac{d\overline{r}}{du}du$$

$$\int_{-L} \overline{F}(\overline{r})\cdot d\overline{r} = \int_b^a \overline{F}(\overline{r}(u))\cdot \frac{d\overline{r}}{du}du = -\int_a^b \overline{F}(\overline{r}(u))\cdot \frac{d\overline{r}}{du}du = -\int_{L} \overline{F}(\overline{r})\cdot d\overline{r}$$

DEFINITION: assume that the curve L consists of a finite number of curves L1, L2,… Then $L=L_1+L_2+\ldots$



THEOREM 2

(6.2 in the textbook)

$$\boxed{\int_{L} \overline{F}(\overline{r})\cdot d\overline{r} = \int_{L_1} \overline{F}(\overline{r})\cdot d\overline{r} + \int_{L_2} \overline{F}(\overline{r})\cdot d\overline{r} + \ldots}$$

Proof:

$$\int_{L_1} \overline{F}(\overline{r})\cdot d\overline{r} + \int_{L_2} \overline{F}(\overline{r})\cdot d\overline{r} = \int_a^b \overline{F}(\overline{r}(u))\cdot \frac{d\overline{r}}{du}du + \int_b^c \overline{F}(\overline{r}(u))\cdot \frac{d\overline{r}}{du}du = \int_a^c \overline{F}(\overline{r}(u))\cdot \frac{d\overline{r}}{du}du = \int_{L} \overline{F}(\overline{r})\cdot d\overline{r}$$

DEFINITION: The line integral of $\overline{A}$ along a closed curve C is called circulation of a $\overline{A}$ along C: *"**cirkulationen**" in swedish*

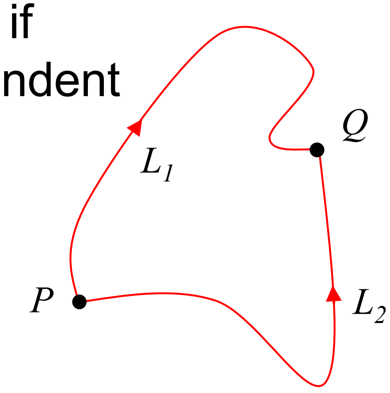
$$\boxed{\oint_C \overline{A}(\overline{r})\cdot d\overline{r}}$$

CONSERVATIVE FIELDS (*konservativt*)

DEFINITION: A vector field $\overline{A}$ is called conservative if: $\oint_C \overline{A}(\overline{r}) \cdot d\overline{r} = 0$ for any C

THEOREM 3 *(6.3 in the textbook)*

The circulation of $\overline{A}$ along all closed curves C is zero if and only if for all points P and Q the line integral of $\overline{A}$ from P to Q is independent from the integration path between P and Q.



PROOF

Assume that L_1 and L_2 are two curves from P to Q.
Then $L=L_1-L_2$ is a closed curve.

(1) The circulation is zero $\Rightarrow$ the line integral from P to Q is independent from the path

$$\left.\begin{aligned} &\oint_L \overline{A}(\overline{r}) \cdot d\overline{r} = 0 \\ &\oint_L \overline{A}(\overline{r}) \cdot d\overline{r} = \int_{L_1-L_2} \overline{A}(\overline{r}) \cdot d\overline{r} = \int_{L_1} \overline{A}(\overline{r}) \cdot d\overline{r} - \int_{L_2} \overline{A}(\overline{r}) \cdot d\overline{r} \end{aligned}\right\} \Longrightarrow \int_{L_1} \overline{A}(\overline{r}) \cdot d\overline{r} = \int_{L_2} \overline{A}(\overline{r}) \cdot d\overline{r}$$

The line integral is independent from the integration path

(2) The line integral from P to Q is independent from the path $\Rightarrow$ the circulation is zero.

$$\left.\begin{aligned} &\int_{L_1} \overline{A}(\overline{r}) \cdot d\overline{r} = \int_{L_2} \overline{A}(\overline{r}) \cdot d\overline{r} \\ &\oint_L \overline{A}(\overline{r}) \cdot d\overline{r} = \int_{L_1-L_2} \overline{A}(\overline{r}) \cdot d\overline{r} = \int_{L_1} \overline{A}(\overline{r}) \cdot d\overline{r} - \int_{L_2} \overline{A}(\overline{r}) \cdot d\overline{r} \end{aligned}\right\} \Longrightarrow \oint_L \overline{A}(\overline{r}) \cdot d\overline{r} = 0$$

The circulation is zero

LINE INTEGRAL and potential

THEOREM 4 *(6.4 in the textbook)*

Consider a curve L described by the parameterization $\overline{r}(u)$ and two points P and Q on L defined by $\overline{r}_P = \overline{r}(p)$ and $\overline{r}_Q = \overline{r}(q)$ then, if $\overline{A} = grad\phi$:

$$\boxed{\int_P^Q \overline{A}(\overline{r}) \cdot d\overline{r} = \phi(q) - \phi(p)}$$

This means that the line integral is independent from the integration path L and depends only on the starting point and on the ending point

PROOF

Using the chain rule for the partial derivative:

$$\int_L \overline{A}(\overline{r}) \cdot d\overline{r} = \int_p^q grad\phi \cdot \frac{d\overline{r}}{du} du = \int_p^q \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) \cdot \left(\frac{dx}{du}, \frac{dy}{du}, \frac{dz}{du} \right) du =$$

$$= \int_p^q \left(\frac{\partial \phi}{\partial x}\frac{dx}{du} + \frac{\partial \phi}{\partial y}\frac{dy}{du} + \frac{\partial \phi}{\partial z}\frac{dz}{du} \right) du = \int_p^q \frac{d}{du}\phi(\overline{r}(u)) du = \phi(q) - \phi(p)$$

Or, easier: $\int_L \overline{A}(\overline{r}) \cdot d\overline{r} = \int_L grad\phi \cdot d\overline{r} = \int_p^q d\phi = \phi(q) - \phi(p)$

OTHER KINDS OF LINE INTEGRALS

- It is possible to combine scalar and vector line elements in many different ways along a curve L and thus get different kinds of line integrals

- Some examples:

$$\int_L \phi(\overline{r})ds$$

$$\int_L \phi(\overline{r})d\overline{r}$$

where $\boxed{d\overline{r} = \hat{e}ds}$

$$\int_L \overline{A}(\overline{r}) \times d\overline{r}$$

Look at Example 6.3 in the book.

- To calculate the integrals:

$$\overline{r} \to \overline{r}(u)$$

$$L \to [a,b] \quad where\, u : a \to b$$

$$d\overline{r} = \frac{d\overline{r}}{du}du \quad or \quad ds = \left|\frac{d\overline{r}}{du}\right|du$$

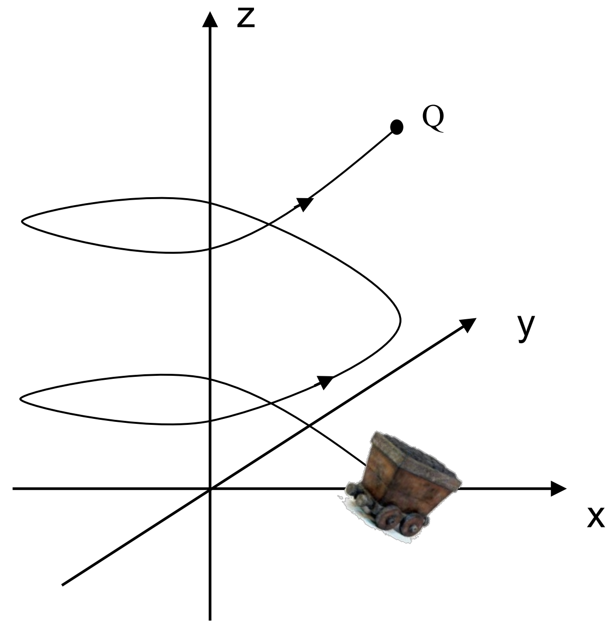
TARGET PROBLEM

(In this example, a Cartesian coordinate system is used)

The force is: $\overline{F} = (-yz, xz, -1)$

The path is L: $\overline{r} = (\cos u, \sin u, u)$ with $u: 0 \to 4\pi$

Calculate the work from P: (1,0,0) till Q: (1,0,4π)



$$W = \int_L \overline{F}(\overline{r}) \cdot d\overline{r} = \int_{a=0}^{b=4\pi} \overline{F}(\overline{r}(u)) \frac{d\overline{r}}{du} du$$

$$\left.\begin{aligned} \overline{F}(\overline{r}(u)) &= (-(\sin u)u, (\cos u)u, -1) \\ \frac{d\overline{r}}{du} &= (-\sin u, \cos u, 1) \end{aligned}\right\} \Rightarrow \overline{F}(\overline{r}(u)) \cdot \frac{d\overline{r}}{du} = (u\sin^2 u + u\cos^2 u - 1) =$$

$$= u(\sin^2 u + \cos^2 u) - 1 = u - 1$$

$$W = \int_L \overline{F}(\overline{r}) \cdot d\overline{r} = \int_0^{4\pi} (u-1) du = \left[\frac{u^2}{2} - u\right]_0^{4\pi} = 8\pi^2 - 4\pi$$

WHICH STATEMENT IS WRONG?

1- The line integral $\int_L \overline{A}(\overline{r}) \times d\overline{r}$ is a vector

2- The line integral $\int \overline{F} \cdot d\overline{r}$ is a scalar

3- The sign of the line integral $\int \overline{F} \cdot d\overline{r}$ depends on the integration path

4- The gradient of a vector field can be written as: $\mathrm{grad}\ \overline{\mathrm{A}}$

PRACTICAL EXAMPLE: THE BIOT-SAVART LAW

The magnetic field in a point P of a steady line current is given by the Biot-Savart law:

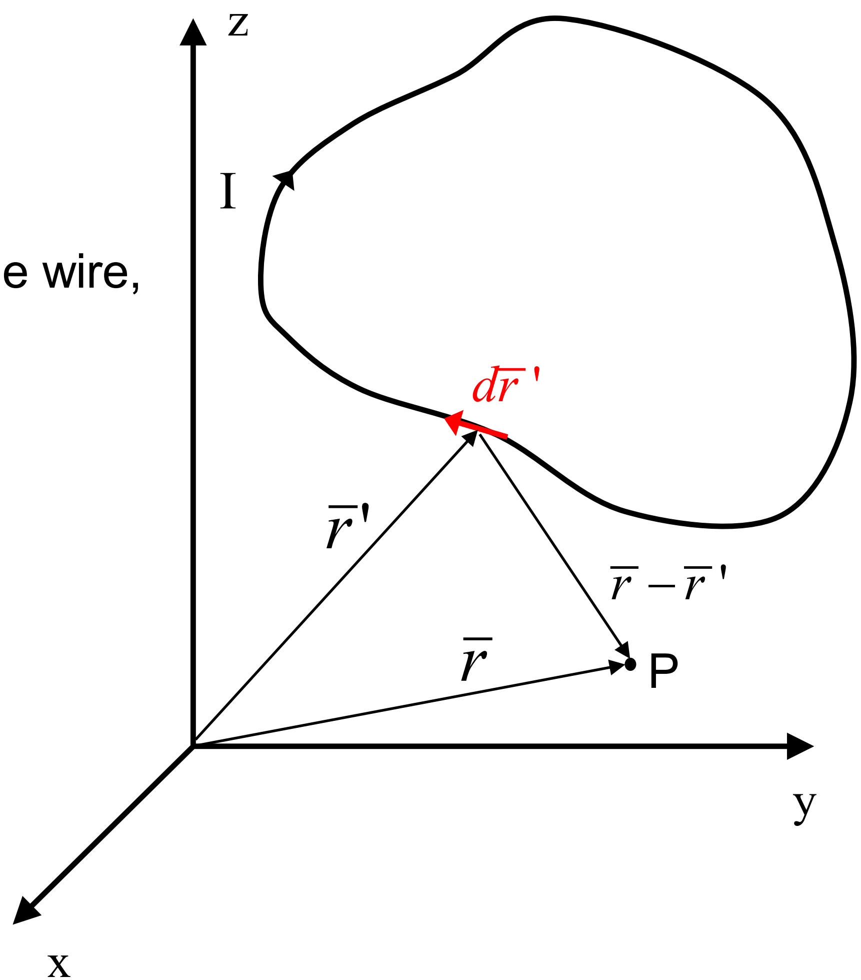
$$\overline{B}(\overline{\mathrm{r}}) = \frac{\mu_0 I}{4\pi} \int_L \frac{d\overline{r}\,' \times \left(\overline{r} - \overline{r}\,'\right)}{\left|\overline{r} - \overline{r}\,'\right|^3}$$

Where $d\overline{r}\,'$ is an infinitesimal length along the wire,

$\overline{r}$ is the position vector of the point P and

$\overline{r}\,'$ is a vector from the origin to $d\overline{r}\,'$

Therefore, $\overline{r} - \overline{r}\,'$ is a vector from $d\overline{r}\,'$ to P



PRACTICAL EXAMPLE: THE BIOT-SAVART LAW

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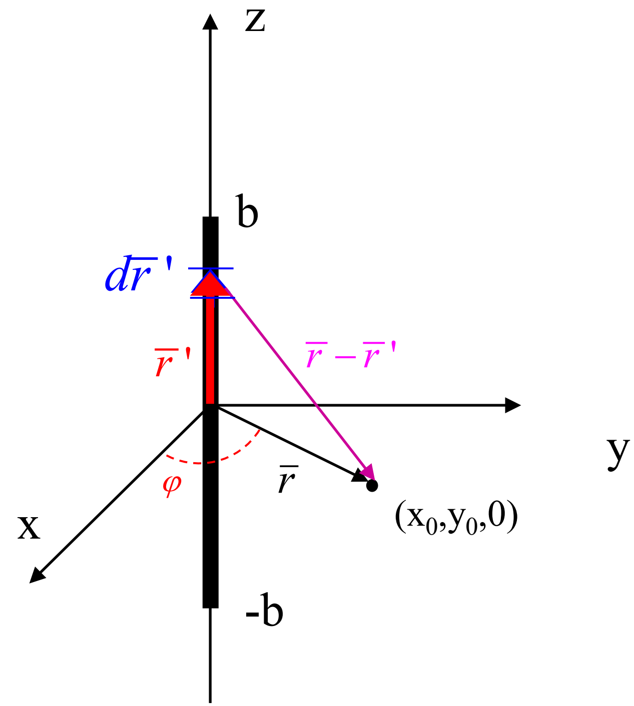
Calculate the magnetic field in $(\mathrm{x}_0,\mathrm{y}_0,0)$ produced by a straight wire with current I and length $2\mathrm{b}$



SOLUTION:

(using a Cartesian coordinate system)

$$\begin{cases} x = 0 \\ y = 0 \\ z = u \end{cases} \Rightarrow \quad \overline{r}\,'(u) = \left(0,0,u\right) \quad \textit{with} \quad u: -b \to +b$$

$$\frac{d\overline{r}\,'}{du}du = \left(0,0,1\right)du = du\,\hat{e}_z$$

$$\overline{r}-\overline{r}\,' = \left(x_0, y_0, -u\right) \quad \Rightarrow \quad \left|\overline{r}-\overline{r}\,'\right| = \sqrt{x_0^2+y_0^2+u^2} = \sqrt{\rho_0^2+u^2}$$

$$\rho_0^2 = x_0^2 + y_0^2 \qquad (\rho_0 \textit{ is the distance of P from the z-axis})$$

$$d\overline{r}\,'\times\left(\overline{r}-\overline{r}\,'\right) = du\,\hat{e}_z\times\left(x_0,y_0,-u\right) = \left(-y_0,x_0,0\right)du$$

$$\overline{B}(\mathrm{r}) = \frac{\mu_0 I}{4\pi}\int_{-b}^{b}\frac{\left(-y_0,x_0,0\right)du}{\left(\rho_0^2+u^2\right)^{3/2}} = \frac{\mu_0 I}{4\pi}\left(-y_0,x_0,0\right)\int_{-b}^{b}\frac{du}{\left(\rho_0^2+u^2\right)^{3/2}} = \frac{\mu_0 I}{4\pi}\left(-y_0,x_0,0\right)\left[\frac{u}{\rho_0^2\sqrt{\rho_0^2+u^2}}\right]_{-b}^{b} = \frac{\mu_0 I}{4\pi}\frac{2b}{\rho_0^2\sqrt{\rho_0^2+b^2}}\left(-y_0,x_0,0\right)$$

$$\left|\overline{B}(\mathrm{r})\right| = \frac{\mu_0 I}{2\pi\rho_0}\frac{b}{\sqrt{\rho_0^2+b^2}}$$

EXERCISE: *Repeat the same calculation using a cylindrical coordinate system*

TARGET PROBLEM

We are making cranberry juice.
After cranberries are squeezed,
It is better to filter the juice!
How much juice flows trough the cloth each second?

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PROBLEM:

Assume that the velocity of the juice is: $\bar{v} = (xy, 0, -z^2)$ and that the surface of the cloth is given by the expressions:

$$S: \begin{cases} z = x^2 + y^2 \\ x^2 + y^2 \leq 1 \\ \hat{n} \cdot \hat{e}_z \leq 0 \end{cases}$$

Calculate the flux of the juice through the surface S.

STEPS to do:

(1) to understand how to calculate the **flux** of a **VECTOR FIELD** $\bar{v}(x, y, z)$

(2) a method to integrate the flux over the whole surface.

THE FLUX

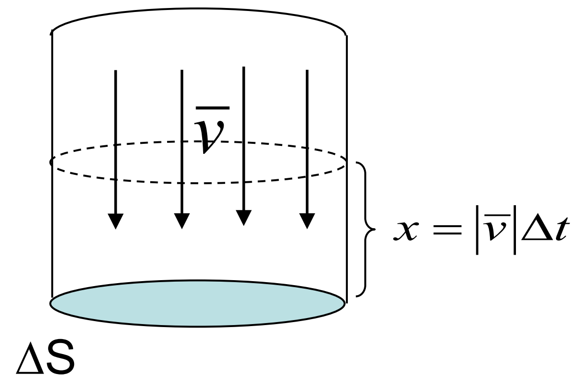
In the juice example, the flux F is the volume of the fluid ΔV that flows through the surface in the time Δt.

$$F = \frac{\Delta V}{\Delta t}$$

STEP 1: - the fluid velocity is perpendicular to the surface
- the surface is not curved

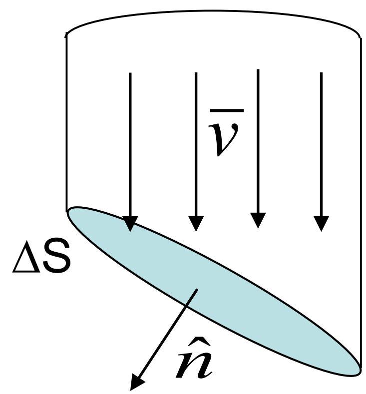
$$\Delta V = x\Delta S = |\overline{v}|\Delta t \Delta S$$

$$F = \frac{\Delta V}{\Delta t} = |\overline{v}|\Delta S$$



STEP 2: - the fluid velocity is NOT perpendicular to the surface
- the surface is not curved

$$F = |\overline{v}_{\perp}|\Delta S = \overline{v}\cdot\hat{n}\Delta S = \overline{v}\cdot\Delta\overline{S}$$

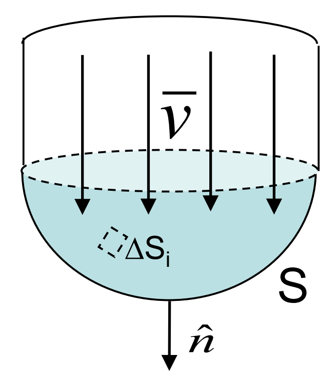


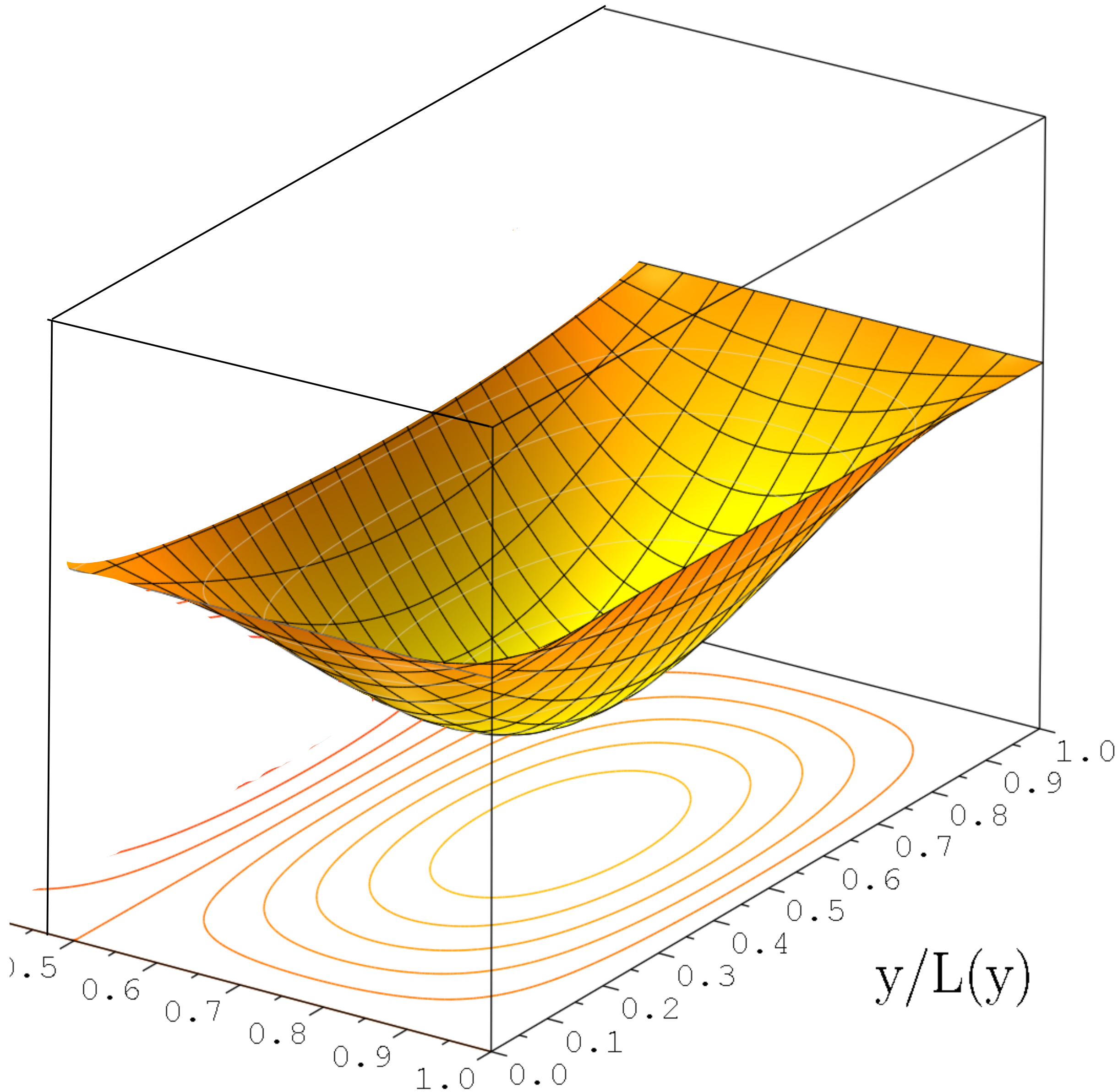
The orientation of the normal depends on the definition of ”positive” and ”negative” side of S

STEP 3: - the surface is curved

$$F = \sum_i F_i = \lim_{\Delta S_i \to 0} \sum_i \overline{v}_i \cdot \Delta\overline{S}_i \equiv \int_S \overline{v}\cdot d\overline{S}$$

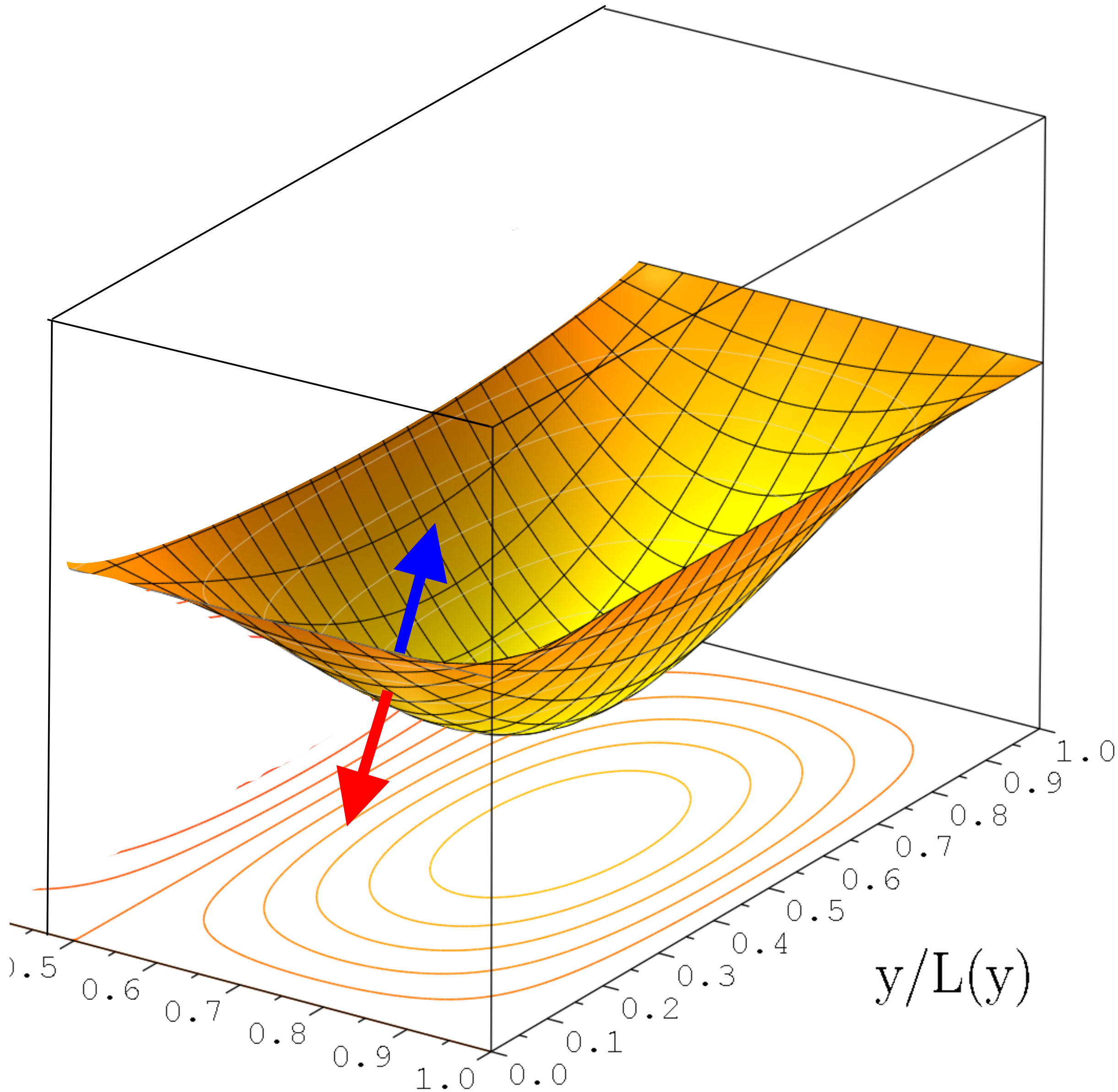
$\boxed{\int_S \overline{v}\cdot d\overline{S}}$ is the flux integral of $\overline{v}$ on the surface S



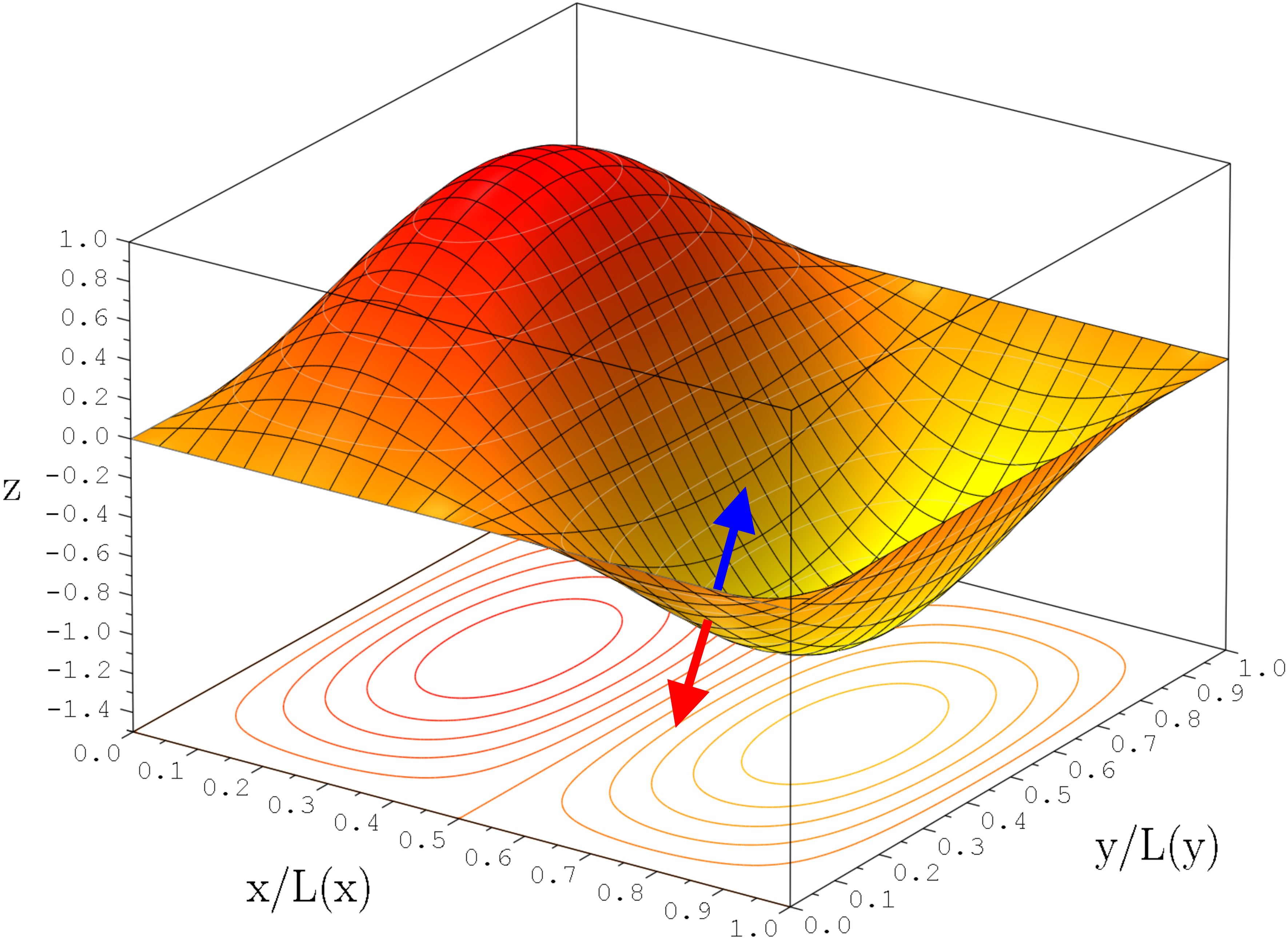


0.5
0.6
0.7
0.8
0.9
1.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
y/L(y)

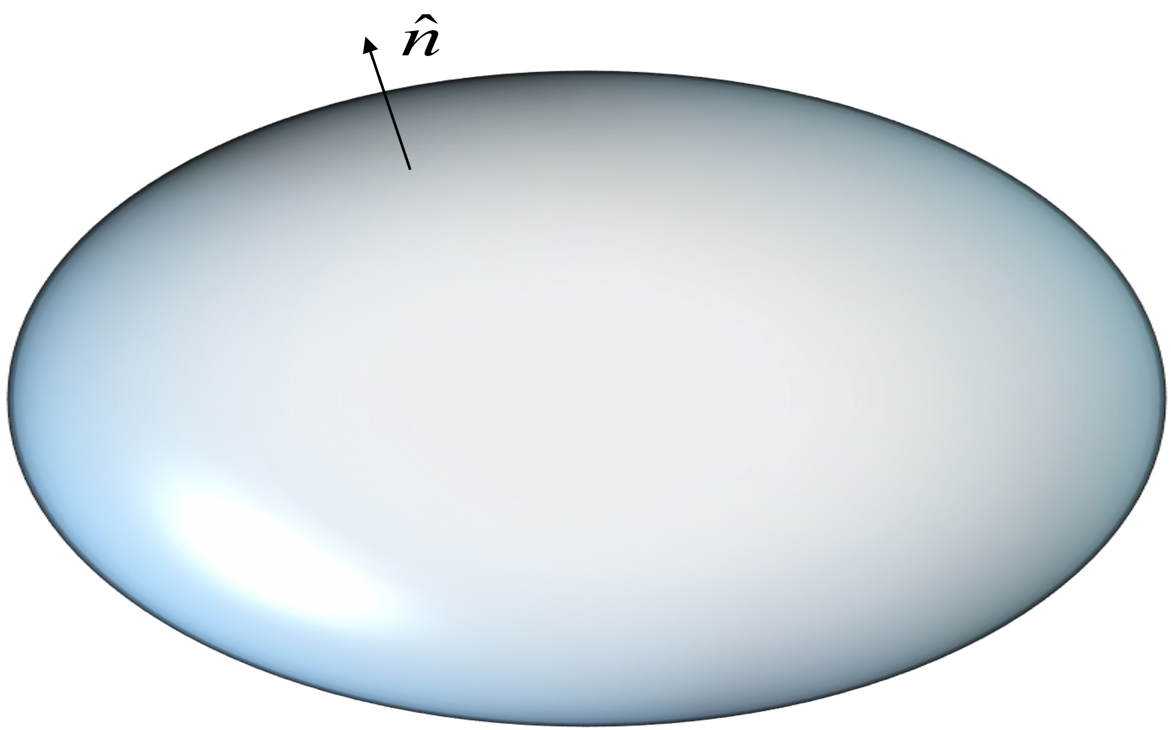
Which is the orientation of the normal to this surface?



Which is the orientation of the normal to this surface?

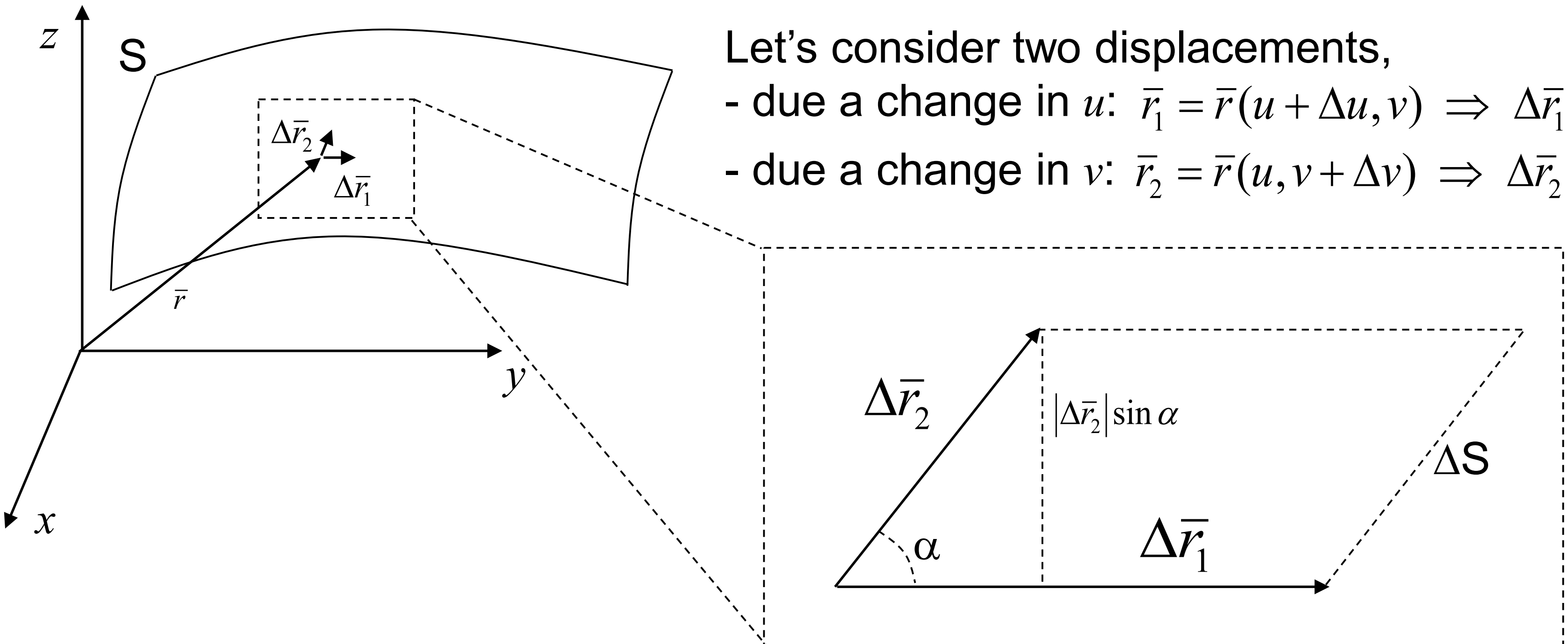


However, when the surface is closed, the usual definition is that the perpendicular to the surface points outwards



$d\bar{S}$ and the FLUX INTEGRAL

- Assume that the surface S is parameterized by $\bar{r} = \bar{r}(u,v)$



Let's consider two displacements,

- due a change in u: $\bar{r}_1 = \bar{r}(u+\Delta u, v) \Rightarrow \Delta\bar{r}_1 = \bar{r}_1 - \bar{r}$
- due a change in v: $\bar{r}_2 = \bar{r}(u, v+\Delta v) \Rightarrow \Delta\bar{r}_2 = \bar{r}_2 - \bar{r}$

The area ΔS is $\Delta S = |\Delta\bar{r}_2| \sin\alpha |\Delta\bar{r}_1| = |\Delta\bar{r}_1 \times \Delta\bar{r}_2|$

$\hat{n}$ is perpendicular to S. But also $\Delta\bar{r}_1 \times \Delta\bar{r}_2$ is perpendicular to S

$$\Rightarrow \Delta\bar{S} = \hat{n}\Delta S = \Delta\bar{r}_1 \times \Delta\bar{r}_2$$

$d\bar{S}$ and the FLUX INTEGRAL

$$d\bar{S} = \lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0}} \Delta\bar{S} = \lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0}} \Delta\bar{r}_1 \times \Delta\bar{r}_2$$

$$d\bar{r}_1 = \lim_{\Delta u \to 0} \Delta\bar{r}_1 = \lim_{\Delta u \to 0} \bar{r}(u+\Delta u, v) - \bar{r}(u,v) =$$

$$= \lim_{\Delta u \to 0} \frac{\bar{r}(u+\Delta u, v) - \bar{r}(u,v)}{\Delta u} \Delta u = \frac{\partial \bar{r}(u,v)}{\partial u} du$$

in the same way:

$$d\bar{r}_2 = \frac{\partial \bar{r}(u,v)}{\partial v} dv$$

$$\Rightarrow \boxed{d\bar{S} = \frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} du dv}$$

So, the flux integral of the vector field $\bar{v}$ on the surface $\bar{S}$ can be calculated as:

$$\boxed{\int_S \bar{v} \cdot d\bar{S} = \int_u \int_v \bar{v}\left(\bar{r}(u,v)\right) \cdot \left(\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}\right) du dv}$$

TARGET PROBLEM

(using a cartesian coordinate system)

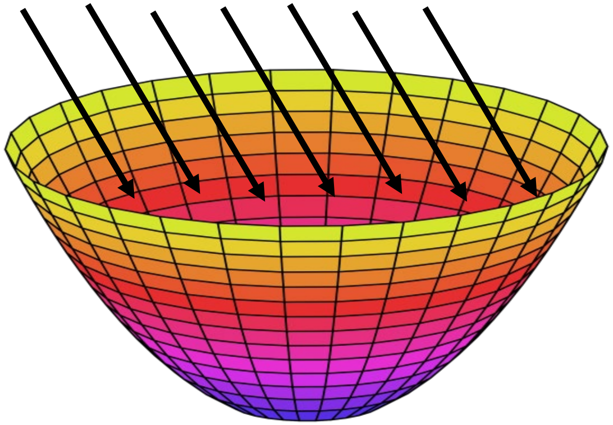
Calculate the flux of the juice.

TARGET PROBLEM

(using a cartesian coordinate system)

Calculate the flux of the juice.
Assume that the velocity is described by the vector field $\overline{v} = (xy, 0, -z^2)$
and that the cloth is describe by the surface S: $\begin{cases} z = x^2 + y^2 \\ x^2 + y^2 \leq 1 \\ \hat{n} \cdot \hat{e}_z < 0 \end{cases}$

This defines the direction of the normal to the surface. It means that the normal has negative z-component.



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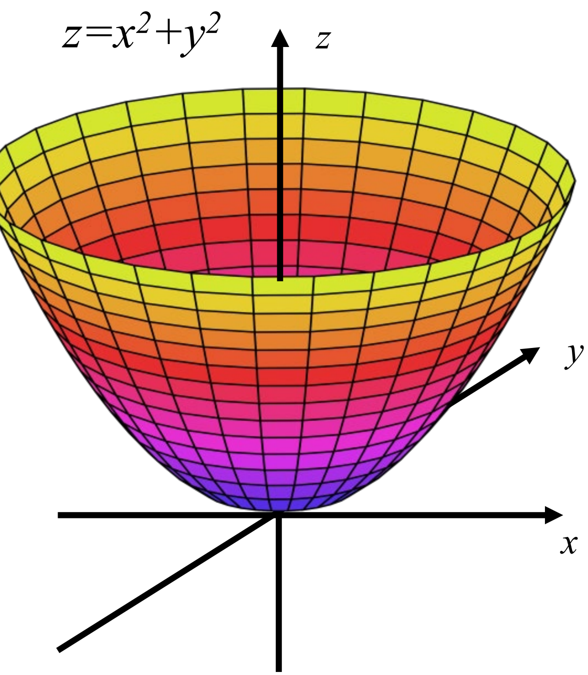
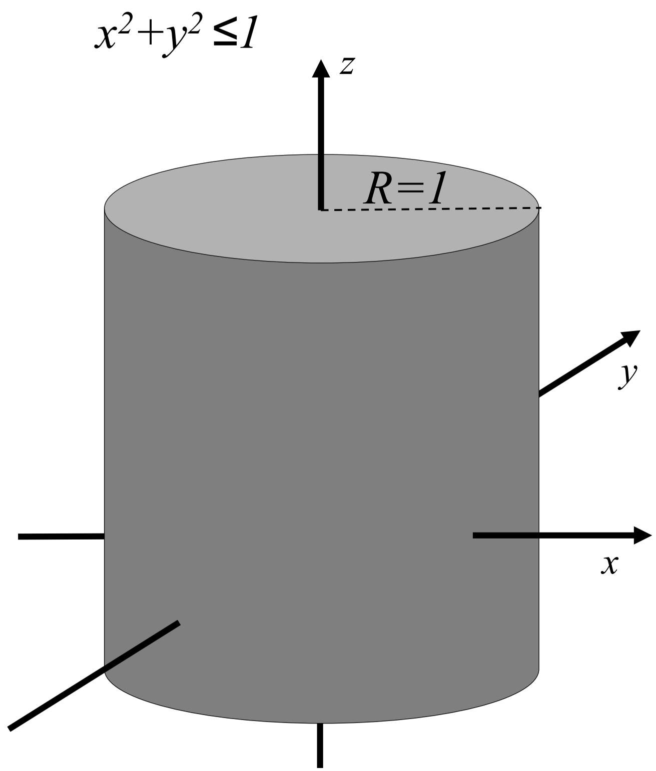
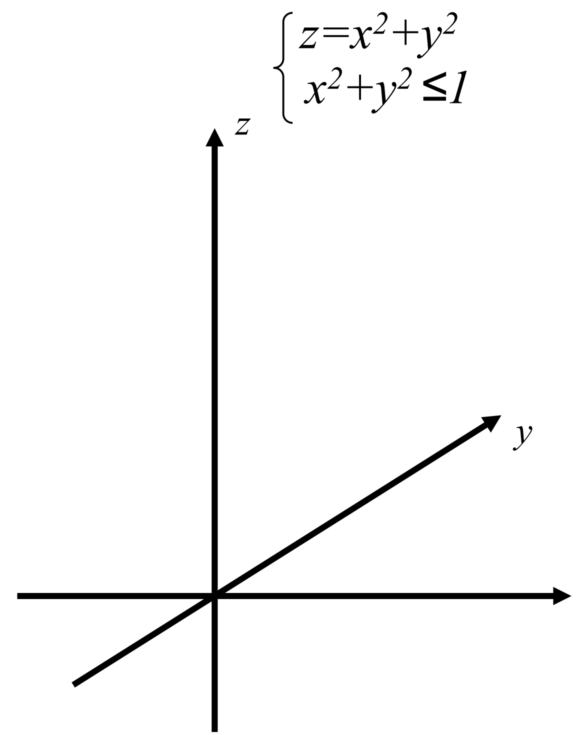
SOLUTION: 1- figure

2- Parameterization of $\bar{S}$

3- Calculate the flux







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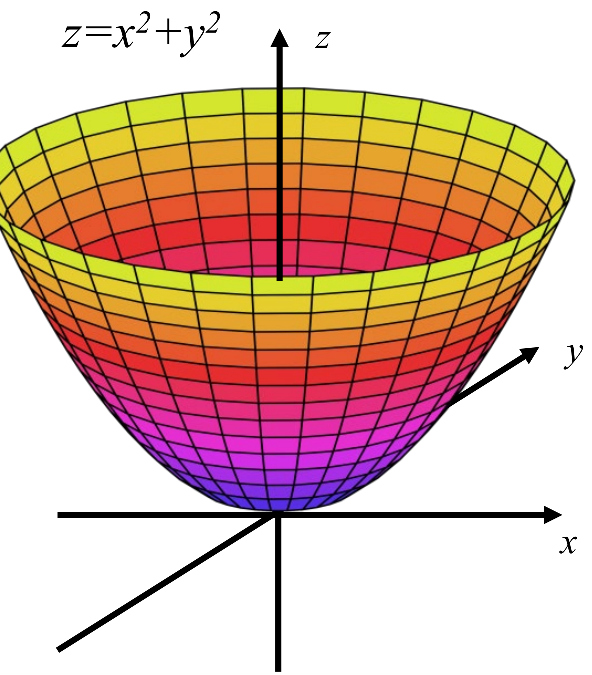

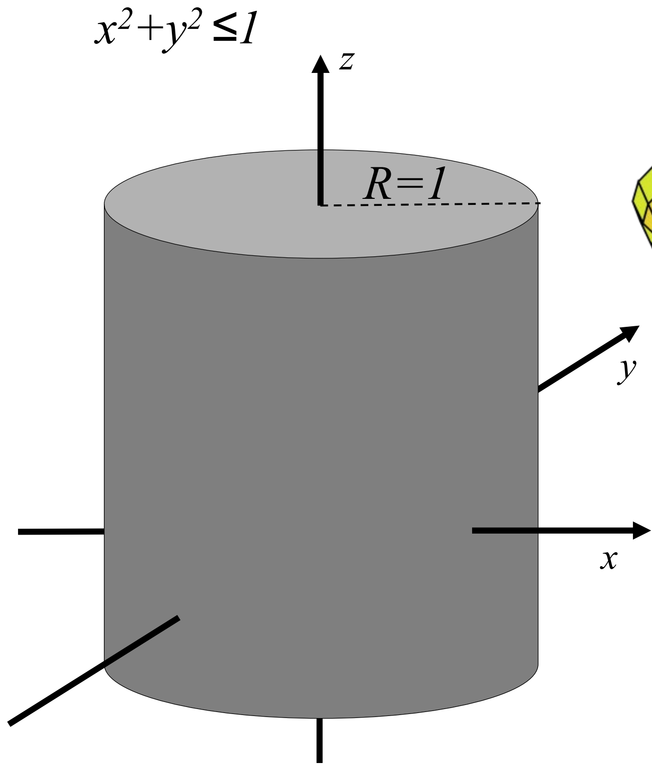

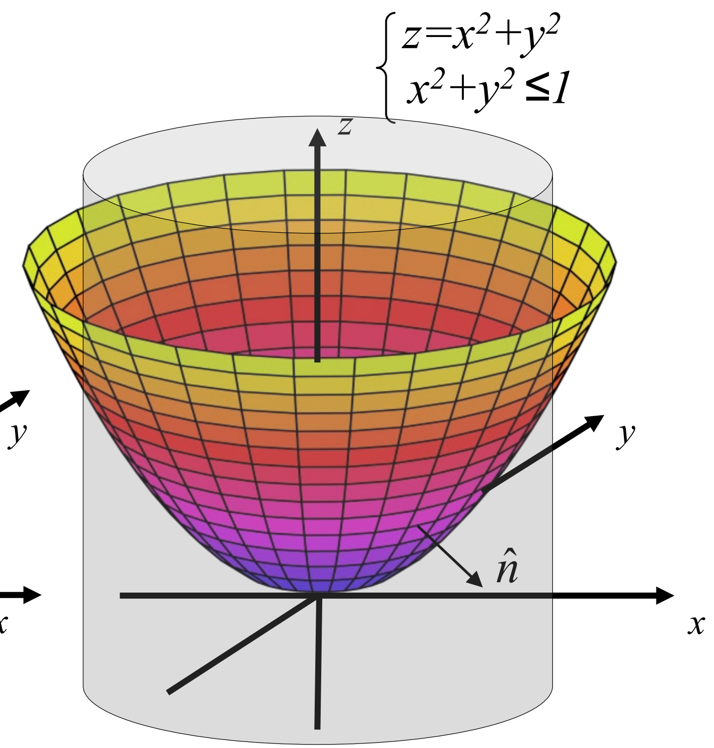

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SOLUTION:
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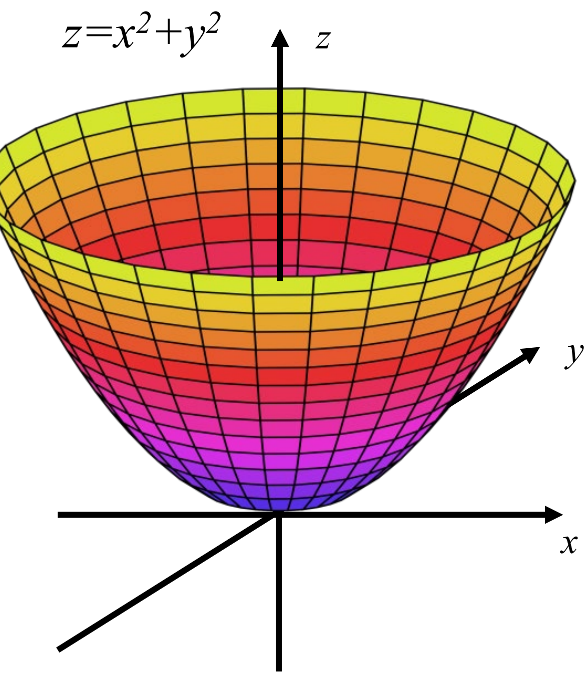
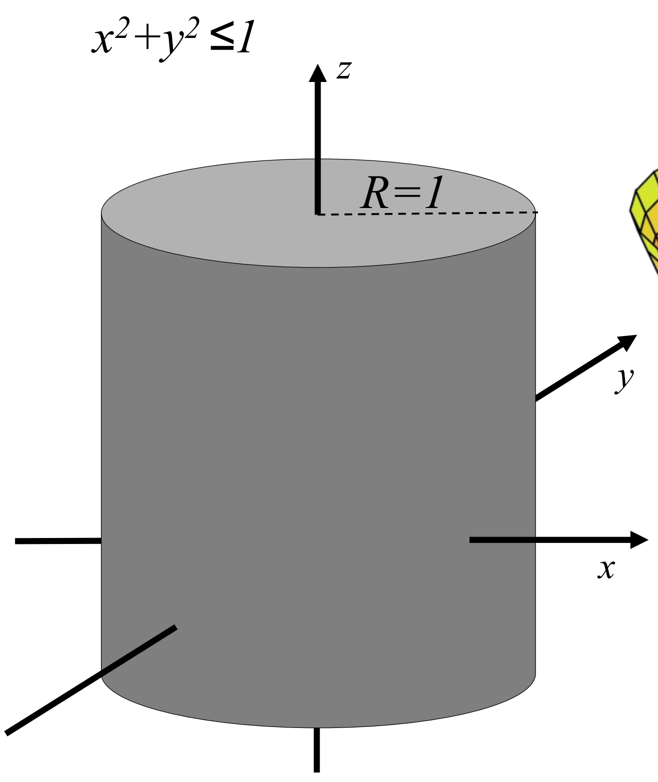
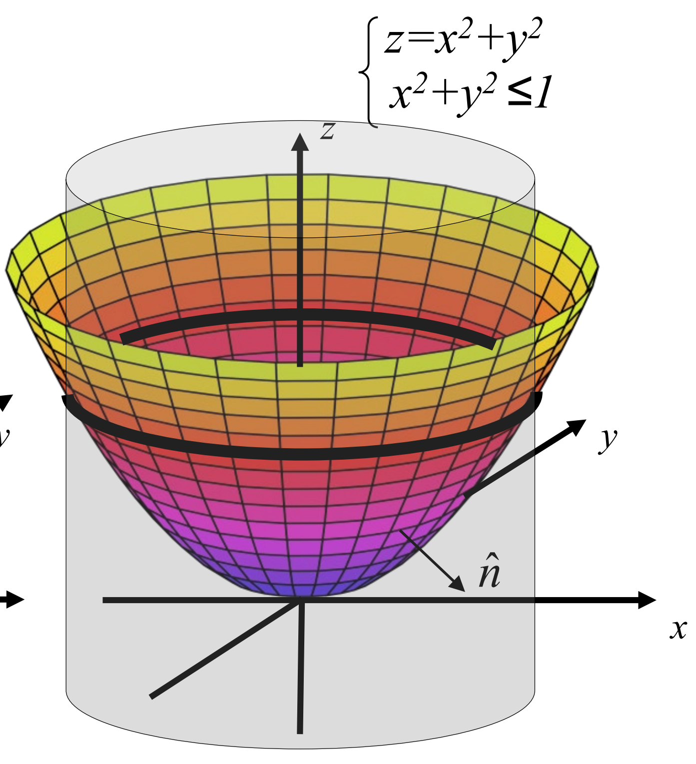
2- Parameterization of $\overline{S}$
3- Calculate the flux







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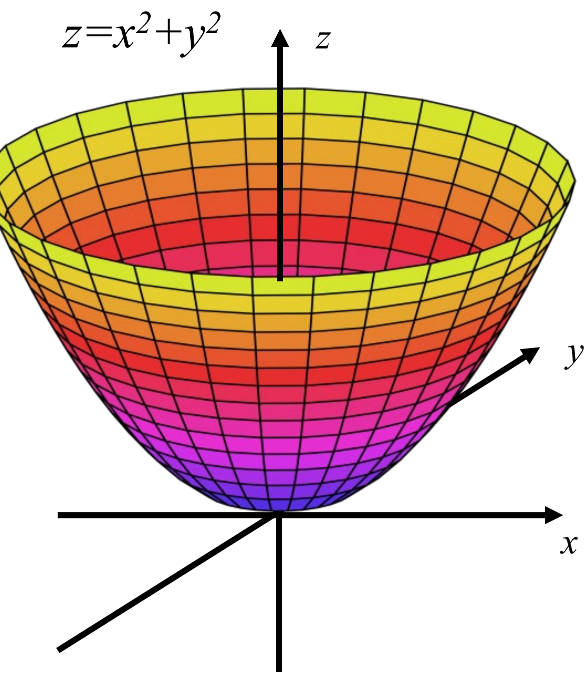
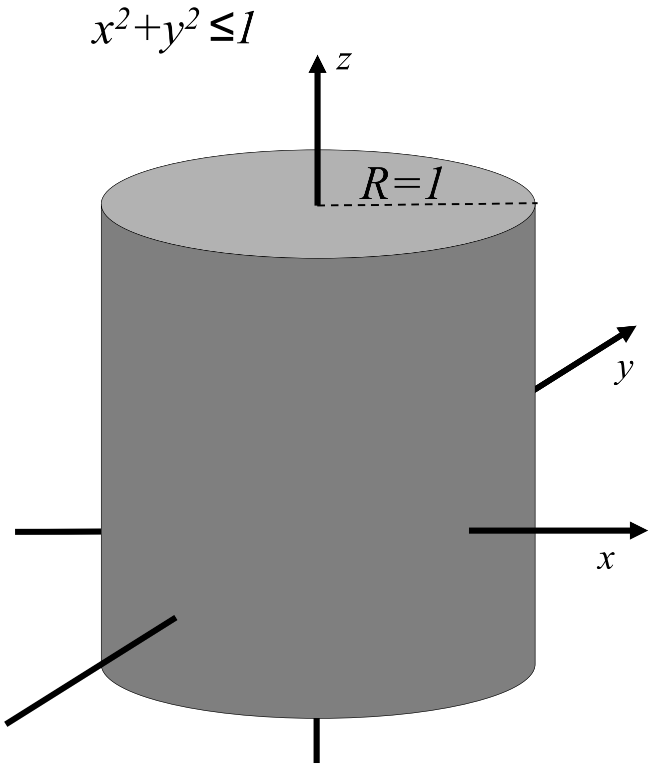
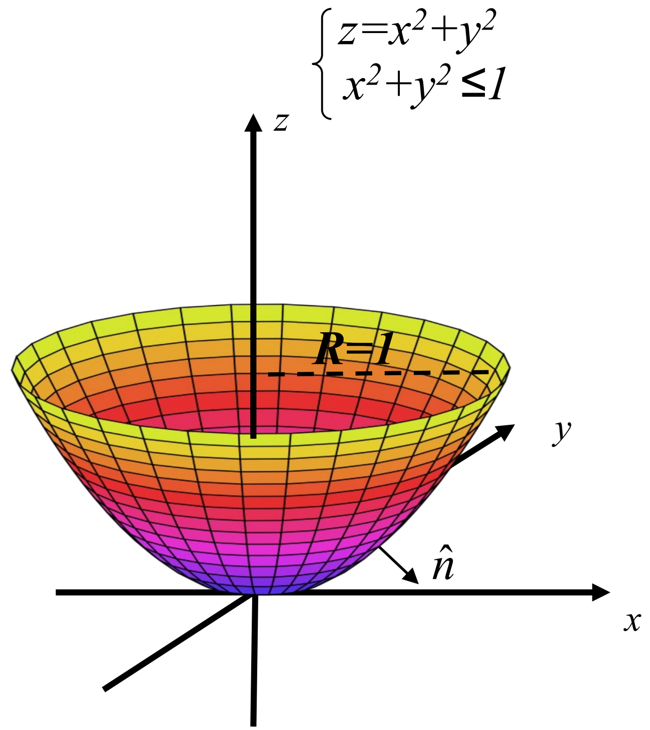
3- Calculate the flux







TARGET PROBLEM

(using a cartesian coordinate system)

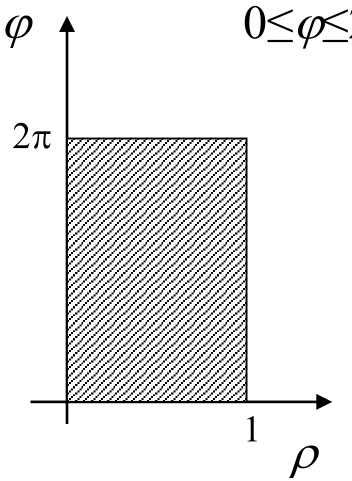
Parameterization of $\overline{S}$ $\quad z=x^2+y^2$

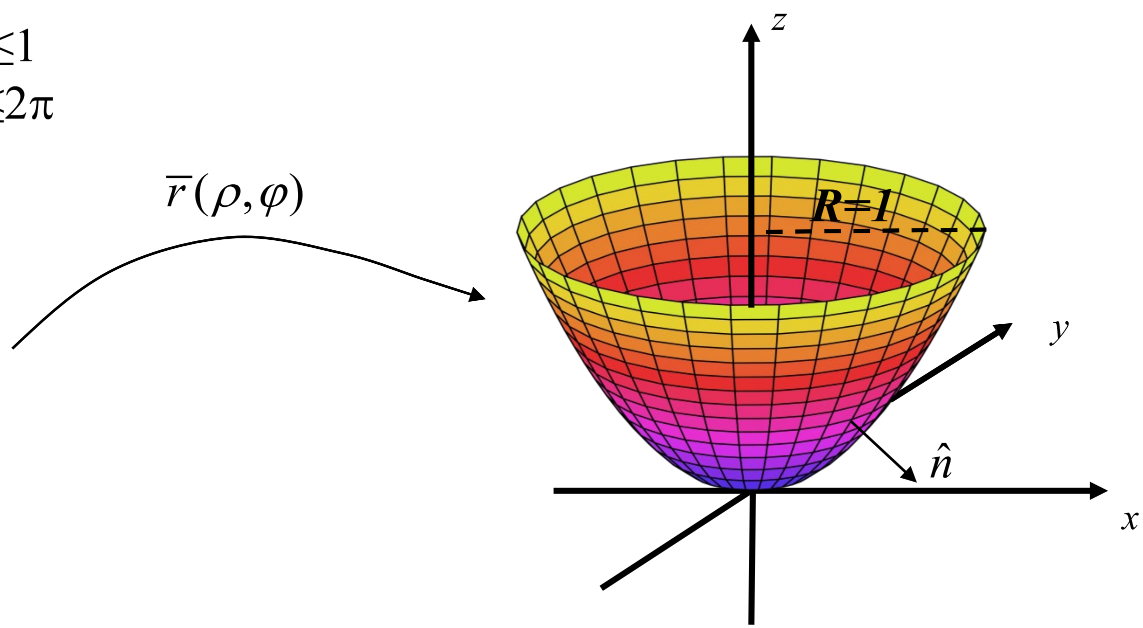
$x^2+y^2 \leq 1$

$$\overline{r}(\rho,\varphi) \quad \begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = x^2 + y^2 = (\rho\sin\varphi)^2 + (\rho\sin\varphi)^2 = \rho^2 \end{cases}$$

$0 \leq \rho \leq 1$

$0 \leq \varphi \leq 2\pi$





Parameterization of the vector field: $\quad \overline{A} = (xy, 0, z^2) = (\rho^2 \sin\varphi\cos\varphi,\ 0,\ -\rho^4)$

TARGET PROBLEM

(using a cartesian coordinate system)

The flux can be calculated as: $F = \iint_S \overline{v} \cdot d\overline{S} = \iint_S \overline{v}(\overline{r}(\rho,\varphi)) \cdot \left(\frac{\partial \overline{r}}{\partial \rho} \times \frac{\partial \overline{r}}{\partial \varphi} \right) d\rho d\varphi$

The term $\left(\frac{\partial \overline{r}}{\partial \rho} \times \frac{\partial \overline{r}}{\partial \varphi} \right)$ can be calculated in this way:

$$\left. \begin{aligned} \frac{\partial \overline{r}}{\partial \rho} &= (\cos\varphi, \sin\varphi, 2\rho) \\ \frac{\partial \overline{r}}{\partial \varphi} &= (-\rho\sin\varphi, \rho\cos\varphi, 0) \end{aligned} \right\} \Rightarrow \left(\frac{\partial \overline{r}}{\partial \rho} \times \frac{\partial \overline{r}}{\partial \varphi} \right) = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \cos\varphi & \sin\varphi & 2\rho \\ -\rho\sin\varphi & \rho\cos\varphi & 0 \end{vmatrix} =$$

$$(-2\rho^2\cos\varphi, -2\rho^2\sin\varphi, \rho\cos^2\varphi + \rho\sin^2\varphi) =$$

$$(-2\rho^2\cos\varphi, -2\rho^2\sin\varphi, \rho)$$

Note that $\left(\frac{\partial \overline{r}}{\partial \rho} \times \frac{\partial \overline{r}}{\partial \varphi} \right)$ has a z-component that is always positive, while according to the definition of the problem, it is supposed to be negative.
How to take care of this?
We have two options:

- We can solve the integral as usual, but then we must change the sign to the final answer
- There is no rule in the order of the parameters, so we could have calculated the term above as: $\left(\frac{\partial \overline{r}}{\partial \varphi} \times \frac{\partial \overline{r}}{\partial \rho} \right) = (2\rho^2\cos\varphi, 2\rho^2\sin\varphi, -\rho)$

TARGET PROBLEM

(using a cartesian coordinate system)

$$\iint_{-S} \overline{v}\cdot d\overline{S} = \iint_{-S} \overline{v}(\overline{r}(\rho,\varphi))\cdot\left(\frac{\partial\overline{r}}{\partial\rho}\times\frac{\partial\overline{r}}{\partial\varphi}\right)d\rho d\varphi =$$

$$\int_0^{2\pi}\int_0^1\left(\rho^2\sin\varphi\,\cos\varphi,0,-\rho^4\right)\cdot(-2\rho^2\cos\varphi,-2\rho^2\sin\varphi,\rho)d\rho d\varphi =$$

$$\int_0^{2\pi}\int_0^1\left(-2\rho^4\sin\varphi\,\cos^2\varphi+0-\rho^5\right)d\rho d\varphi =$$

$$\int_0^{2\pi}\left[-\frac{2}{5}\rho^5\sin\varphi\,\cos^2\varphi-\frac{1}{6}\rho^6\right]_0^1 d\varphi = \int_0^{2\pi}\left(-\frac{2}{5}\sin\varphi\,\cos^2\varphi-\frac{1}{6}\right)d\varphi =$$

$$\left[-\frac{2}{5}\left(-\frac{\cos^3\varphi}{3}\right)-\frac{1}{6}\varphi\right]_0^{2\pi} = -\frac{\pi}{3}$$

But we must change sign! The answer is $+\dfrac{\pi}{3}$

WHICH STATEMENT IS WRONG?

1- The flux integral is a scalar

2- Flux integrals can be calculated also on a closed surface.

3- The normal to the integration surface
always points "downwards" (in the negative direction of the z-axis)

4- The flux through a membrane can be
calculated with flux integrals.

FLUX OF THE ELECTRIC FIELD PRODUCED BY A POINT CHARGE

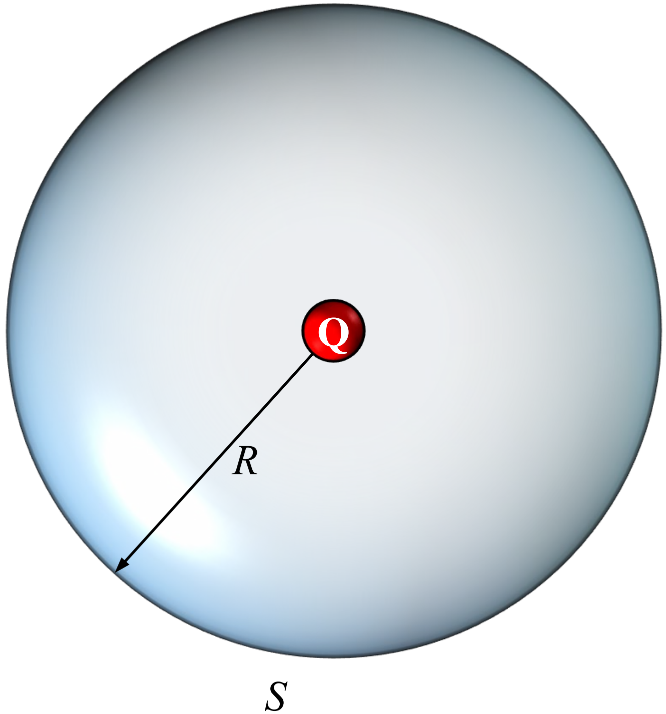
The electric field produced by a point charge located in the origin is:

$$\bar{E} = \frac{Q}{4\pi\varepsilon_0} \frac{x\hat{e}_x + y\hat{e}_y + z\hat{e}_z}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

using a Cartesian coordinate system

$$\bar{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{e}_r$$

using a spherical coordinate system



Calculate the flux of the electric field on a sphere S centred in the origin and with radius R using:

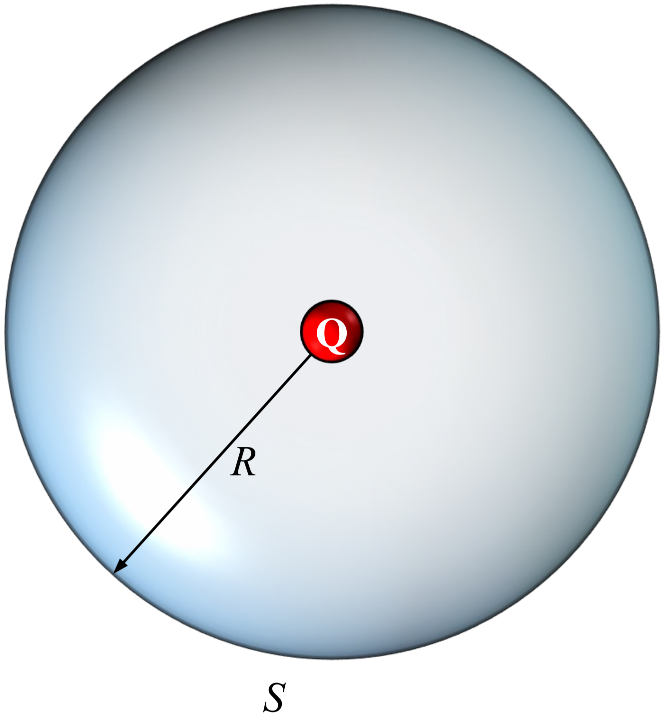
(a) a Cartesian coordinate system

(b) a spherical coordinate system

(a) Using a Cartesian coordinate system

$$\overline{E} = \frac{Q}{4\pi\varepsilon_0} \frac{x\hat{e}_x + y\hat{e}_y + z\hat{e}_z}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\int_S \overline{E} \cdot d\overline{S} = \int_u \int_v \overline{E}\left(\overline{r}(u,v)\right) \cdot \left(\frac{\partial \overline{r}}{\partial u} \times \frac{\partial \overline{r}}{\partial v} \right) du dv$$



Which parameterization to use?
The surface is a sphere, so a smart choice is to use spherical coordinates:

$$\begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases}$$

with: $0 \le \theta \le \pi$
$0 \le \varphi \le 2\pi$

IMPORTANT:
(1) to use spherical coordinates $r,\ \theta,\ \varphi$ does not imply that a spherical coordinate system is used!
The coordinate system is still Cartesian as long as the componenents of a vector are the projections along the x-axis y-axis and z-axes.
(2) The flux is calculated on the surface of the sphere with radius R (which is constant).
So no integration in r is necessary. The integration is only in $\theta,\ \varphi$.
(3) Each point on the sphere has distance R from the origin. So, on the surface of S: $x^2+y^2+z^2=R^2$

$$\int_S \overline{E} \cdot d\overline{S} = \int_\theta \int_\varphi \overline{E}\left(\overline{r}(\theta,\varphi)\right) \cdot \left(\frac{\partial \overline{r}}{\partial \theta} \times \frac{\partial \overline{r}}{\partial \varphi} \right) d\theta d\varphi$$

Parameterization of the sphere:

$$\overline{r}(\theta,\varphi) = (R\sin\theta\cos\varphi, R\sin\theta\sin\varphi, R\cos\theta) = R\sin\theta\cos\varphi\hat{e}_x + R\sin\theta\sin\varphi\hat{e}_y + R\cos\theta\hat{e}_z$$

$$\int_S \overline{E}\cdot d\overline{S} = \int_\theta \int_\varphi \overline{E}\left(\overline{r}(\theta,\varphi)\right)\cdot\left(\frac{\partial \overline{r}}{\partial\theta}\times\frac{\partial \overline{r}}{\partial\varphi}\right)d\theta d\varphi$$

$$\overline{r}(\theta,\varphi) = (R\sin\theta\cos\varphi, R\sin\theta\sin\varphi, R\cos\theta) = R\sin\theta\cos\varphi\hat{e}_x + R\sin\theta\sin\varphi\hat{e}_y + R\cos\theta\hat{e}_z$$

$$\frac{\partial \overline{r}}{\partial\theta} = R\cos\theta\cos\varphi\hat{e}_x + R\cos\theta\sin\varphi\hat{e}_y - R\sin\theta\hat{e}_z$$

$$\frac{\partial \overline{r}}{\partial\varphi} = -R\sin\theta\sin\varphi\hat{e}_x + R\sin\theta\cos\varphi\hat{e}_y$$

$$\left(\frac{\partial \overline{r}}{\partial\theta}\times\frac{\partial \overline{r}}{\partial\varphi}\right) = R\sin\theta\left(\mathrm{R}\sin\theta\cos\varphi\hat{e}_x + R\sin\theta\sin\varphi\hat{e}_y + R\cos\theta\hat{e}_z\right) = R\sin\theta\left(x\hat{e}_x + y\hat{e}_y + z\hat{e}_z\right)$$

$$\overline{E}\left(\overline{r}(\theta,\varphi)\right)\cdot\left(\frac{\partial \overline{r}}{\partial\theta}\times\frac{\partial \overline{r}}{\partial\varphi}\right) = \frac{Q}{4\pi\varepsilon_0}R\sin\theta\frac{x\hat{e}_x + y\hat{e}_y + z\hat{e}_z}{\left(x^2+y^2+z^2\right)^{3/2}}\cdot\left(x\hat{e}_x + y\hat{e}_y + z\hat{e}_z\right) =$$

Remember that $x^2+y^2+z^2 = R^2$

$$= \frac{Q}{4\pi\varepsilon_0}R\sin\theta\frac{x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^{3/2}} = \frac{Q}{4\pi\varepsilon_0}R\sin\theta\frac{1}{\left(x^2+y^2+z^2\right)^{1/2}} = \frac{Q}{4\pi\varepsilon_0}\sin\theta$$

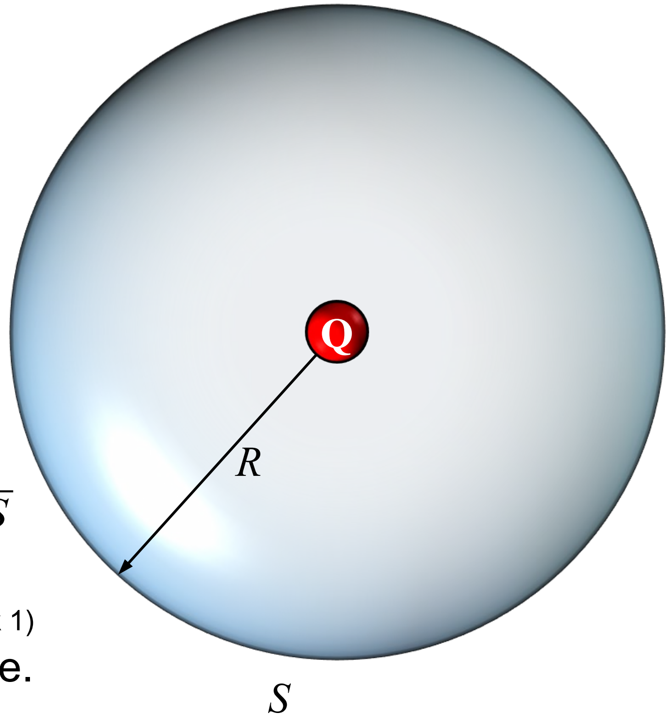
$$\int_S \overline{E}\cdot d\overline{S} = \int_\theta\int_\varphi \frac{Q}{4\pi\varepsilon_0}\sin\theta d\theta d\varphi = \frac{Q}{4\pi\varepsilon_0}\int_{\theta=0}^{\theta=\pi}\int_{\varphi=0}^{\varphi=2\pi}\sin\theta d\theta d\varphi = \frac{Q}{\varepsilon_0}$$

(b) Using a spherical coordinate system

$$\overline{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{e}_r$$

We need to calculate: $\int_S \overline{E} \cdot d\overline{S}$

In spherical coordinate system, it is much easier.
First we need to express $d\overline{S}$ in a spherical coordinate system:
we need to calculate the absolute value and the direction of $d\overline{S}$



- absolute value: $\left|d\overline{S}\right| = r^2 \sin\theta d\varphi d\theta$ (see the slides on "vector algebra", week 1)
- direction: it must be perpendicular to the surface of the sphere.
  The radial direction is perpendicular to S.

$$d\overline{S} = r^2 \sin\theta d\varphi d\theta \; \hat{e}_r$$

Remember that $\hat{e}_r \cdot \hat{e}_r = 1$

$$\int_S \overline{E} \cdot d\overline{S} = \int_S \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{e}_r \cdot (r^2 \sin\theta) d\varphi d\theta \; \hat{e}_r = \frac{Q}{4\pi\varepsilon_0}\int_S (\sin\theta) d\varphi d\theta =$$

$$= \frac{Q}{4\pi\varepsilon_0}\int_{\theta=0}^{\theta=\pi}\int_{\varphi=0}^{\varphi=2\pi} \sin\theta d\theta d\varphi = \frac{Q}{\varepsilon_0}$$