VEKTORANALYS

HT 2021

CELTE / CENMI

ED1110

LINE INTEGRALS and FLUX INTEGRALS: ÖVNINGAR

Kursvecka 2

Kapitel 6-7 (*Vektoranalys*, 1:e uppl, Frassinetti/Scheffel)

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PROBLEM 1

Calculate the potential ϕ for the following vector field: $\overline{A} = 2xz\hat{e}_x + 2z^2\hat{e}_y + \left(x^2 + 4yz\right)\hat{e}_z$

SOLUTION

Step 1. Verify that the vector field can have a potential: $\dfrac{\partial A_x}{\partial y} = \dfrac{\partial A_y}{\partial x} \qquad \dfrac{\partial A_y}{\partial z} = \dfrac{\partial A_z}{\partial y} \qquad \dfrac{\partial A_z}{\partial x} = \dfrac{\partial A_x}{\partial z}$

Theorem 5.1

$$\left.\begin{array}{lll}
\dfrac{\partial A_x}{\partial y} = 0 & \dfrac{\partial A_y}{\partial x} = 0 & \Rightarrow \dfrac{\partial A_x}{\partial y} = \dfrac{\partial A_y}{\partial x} \\
\dfrac{\partial A_y}{\partial z} = 4z & \dfrac{\partial A_z}{\partial y} = 4z & \Rightarrow \dfrac{\partial A_y}{\partial z} = \dfrac{\partial A_z}{\partial y} \\
\dfrac{\partial A_z}{\partial x} = 2x & \dfrac{\partial A_x}{\partial z} = 2x & \Rightarrow \dfrac{\partial A_z}{\partial x} = \dfrac{\partial A_x}{\partial z}
\end{array}\right\} \Rightarrow \overline{A} \quad \textit{might have a potential}$$

Step 2. Calculate the potential

$$\overline{A} = (2xz, 2z^2, x^2 + 4yz) \qquad (a)$$

$$\overline{A} = grad\,\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right) \qquad (b)$$

$$\frac{\partial \phi}{\partial x} = 2xz \quad \Rightarrow \quad \phi = x^2 z + F(y,z) \qquad (c)$$

$$\left.\begin{array}{ll}
\textit{From (a) and (b)} & \dfrac{\partial \phi}{\partial y} = 2z^2 \\
\textit{From (c)} & \dfrac{\partial \phi}{\partial y} = \dfrac{\partial F(y,z)}{\partial y}
\end{array}\right\} \Rightarrow \frac{\partial F(y,z)}{\partial y} = 2z^2 \quad \Rightarrow \quad F(y,z) = 2yz^2 + G(z)$$

$$\phi = x^2 z + 2yz^2 + G(z) \qquad (d)$$

$$\left.\begin{array}{ll}
\textit{From (a) and (b)} & \dfrac{\partial \phi}{\partial z} = x^2 + 4yz \\
\textit{From (d)} & \dfrac{\partial \phi}{\partial z} = x^2 + 4yz + \dfrac{\partial G(z)}{\partial z}
\end{array}\right\} \Rightarrow \frac{\partial G(z)}{\partial z} = 0 \quad \Rightarrow \quad G(z) = const.$$

$$\boxed{\phi = x^2 z + 2yz^2 + const.}$$

PROBLEM 2

Calculate the line integral of the vector field: $\overline{A} = (2-y)\hat{e}_x - xy\hat{e}_y + \hat{e}_z$

along: (A) the curve *L* $\begin{cases} 4x - y^2 = 0 \\ x^2 + y^2 - z = 0 \end{cases}$

from the point (0,0,0) to the point (1,2,5)

(B) a straight line from point (0,0,0) to (1,2,0) and then from (1,2,0) to (1,2,5)

SOLUTION to problem 2 (part A)

The line integral is: $\int_L \overline{A}(\overline{r}) \cdot d\overline{r} = \int_a^b \overline{A}(\overline{r}(u)) \cdot \frac{d\overline{r}}{du} du$

STEP 1: we need to find a parameterization of the curve L in order to have $\overline{r} = \overline{r}(u)$

If $u=y$ we have:

$$x(u) = \frac{u^2}{4} \quad y(u) = u \quad z(u) = \frac{u^4}{16} + u^2 \qquad \overline{r}(u) = \left(\frac{u^2}{4}, u, \frac{u^4}{16} + u^2\right)$$

(See problem 1 week 1)

from point (0,0,0) till the point (1,2,5) $\Rightarrow$ u: a$\rightarrow$b with a=0 and b=2

STEP 2: we calculate the integral

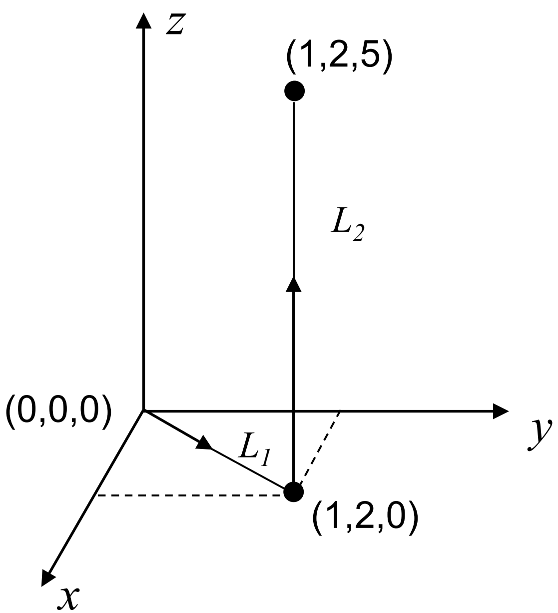
$$\left.\begin{aligned} \overline{A}(\overline{r}(u)) &= (2 - y(u), -x(u)y(u), 1) = \left(2-u, -\frac{u^3}{4}, 1\right) \\ \frac{d\overline{r}}{du} &= \left(\frac{2u}{4}, 1, \frac{4u^3}{16} + 2u\right) = \left(\frac{u}{2}, 1, \frac{u^3}{4} + 2u\right) \end{aligned}\right\} \Rightarrow$$

$$\int_L \overline{A}(\overline{r}) \cdot d\overline{r} = \int_0^2 \left(2-u, -\frac{u^3}{4}, 1\right) \cdot \left(\frac{u}{2}, 1, \frac{u^3}{4} + 2u\right) du =$$

$$\int_0^2 \left(u - \frac{u^2}{2} - \frac{u^3}{4} + \frac{u^3}{4} + 2u\right) du = \left[3\frac{u^2}{2} - \frac{u^3}{6}\right]_0^2 = \frac{14}{3}$$

SOLUTION to problem 2 (part B)

straight line $\underbrace{\text{from point (0,0,0) till (1,2,0)}}_{L_1}$ and $\underbrace{\text{then from (1,2,0) till (1,2,5)}}_{L_2}$ $\Rightarrow \int_L \overline{A}(\overline{r})\cdot d\overline{r} = \int_{L_1} \overline{A}(\overline{r})\cdot d\overline{r} + \int_{L_2} \overline{A}(\overline{r})\cdot d\overline{r}$



- We start with the first integral.

STEP 1: parameterization of L_1 : $\overline{r}(u) = (u, 2u, 0)$

u: $0 \rightarrow 1$

$$\frac{d\overline{r}}{du} = (1,2,0)$$

STEP 2: line integral calculation

$$\int_{L_1} \overline{A}(\overline{r})\cdot d\overline{r} = \int_a^b \overline{A}(\overline{r}(u))\cdot \frac{d\overline{r}}{du}du = \int_0^1 (2-2u, -2u^2, 1)\cdot(1,2,0)du =$$

$$\int_0^1 (2-2u-4u^2)du = \left[2u - u^2 - \frac{4}{3}u^3\right]_0^1 = \left(2-1-\frac{4}{3}\right) = -\frac{1}{3}$$

- We continue with the second integral.

STEP 1: parameterization of L_2 : $\overline{r}(u) = (1, 2, u)$

u: $0 \rightarrow 5$

$$\frac{d\overline{r}}{du} = (0,0,1)$$

STEP 2: line integral calculation

$$\int_{L_2} \overline{A}(\overline{r})\cdot d\overline{r} = \int_a^b \overline{A}(\overline{r}(u))\cdot \frac{d\overline{r}}{du}du = \int_0^5 (2-2, -2, 1)\cdot(0,0,1)du = \int_0^5 du = 5$$

- Finally: $$\int_L \overline{A}(\overline{r})\cdot d\overline{r} = \int_{L_1} \overline{A}(\overline{r})\cdot d\overline{r} + \int_{L_2} \overline{A}(\overline{r})\cdot d\overline{r} = -\frac{1}{3} + 5 = \frac{14}{3}$$

PROBLEM 3

Calculate the line integral of the vector field: $\overline{A}=\frac{x}{\sqrt{x^2+y^2+5z^2}}\hat{e}_x+\frac{\sqrt{5}yz}{\sqrt{x^2+y^2+z^2}}\hat{e}_y+\frac{e^x\sin y}{1+\ln(xyz)}\hat{e}_z$

Along the path L defined by:

$$L=\begin{cases}x^2+y^2=4\\ z=1\end{cases}$$

From the point P_1: (2,0,1) to the point P_2: (0,2,1)

SOLUTION

Parameterization of L:

$$\left.\begin{array}{l}x=2\cos\varphi\\ y=2\sin\varphi\\ z=1\end{array}\right\}\quad\Rightarrow\quad L:\ \ \overline{r}(\varphi)=(2\cos\varphi,2\sin\varphi,1)\quad\Rightarrow\quad\frac{d\overline{r}(\varphi)}{d\varphi}=(-2\sin\varphi,2\cos\varphi,0)$$

$$\text{P1}=(2,0,1)\quad\Rightarrow\quad\varphi_1=0$$

$$\text{P2}=(0,2,1)\quad\Rightarrow\quad\varphi_2=\frac{\pi}{2}$$

$$\int_L\overline{A}\cdot d\overline{r}=\int_{\varphi_1}^{\varphi_2}\overline{A}(\overline{r}(\varphi))\cdot\frac{d\overline{r}}{d\varphi}d\varphi$$

$$\overline{A}=\left(\frac{x}{\sqrt{x^2+y^2+5z^2}},\ \frac{\sqrt{5}yz}{\sqrt{x^2+y^2+z^2}},\ \frac{e^x\sin y}{1+\ln(xyz)}\right)=\left(\frac{2\cos\varphi}{\sqrt{(2\cos\varphi)^2+(2\sin\varphi)^2+5(1)^2}},\ \frac{2\sin\varphi\sqrt{5}}{\sqrt{(2\cos\varphi)^2+(2\sin\varphi)^2+(1)^2}},\ \frac{e^x\sin y}{1+\ln(xyz)}\right)=\left(\frac{2\cos\varphi}{3},\ 2\sin\varphi,\ \frac{e^x\sin y}{1+\ln(xyz)}\right)$$

$$\int_L\overline{A}\cdot d\overline{r}=\int_0^{\pi/2}\left(\frac{2\cos\varphi}{3},\ 2\sin\varphi,\ \frac{e^x\sin y}{1+\ln(xyz)}\right)\cdot(-2\sin\varphi,2\cos\varphi,0)\,d\varphi=\int_0^{\pi/2}\left(-\frac{4}{3}\sin\varphi\cos\varphi+4\sin\varphi\cos\varphi\right)d\varphi=\int_0^{\pi/2}\left(\frac{8}{3}\sin\varphi\cos\varphi\right)d\varphi=$$

$$\int_0^{\pi/2}\left(\frac{4}{3}\sin 2\varphi\right)d\varphi=-\frac{4}{3}\left[\frac{\cos 2\varphi}{2}\right]_0^{\pi/2}=-\frac{2}{3}[-1-1]=\frac{4}{3}$$

PROBLEM 4

Calculate the line integral of the vector field: $\overline{A} = x^2\hat{e}_x + \sinh(yz)\hat{e}_y + z^2\hat{e}_z$

$$\left(\sinh x = \frac{e^x - e^{-x}}{2}\right)$$

Along the path L defined by:

$$L = \begin{cases} x^2 - z = 0 \\ y = 2 \end{cases}$$

From the point P_1: (-1,2,1) to the point P_2: (1,2,1).

SOLUTION

Parameterization of L:

$$\left.\begin{array}{l} x = u \\ y = 2 \\ z = u^2 \end{array}\right\} \Rightarrow \quad L: \quad \overline{r}(u) = \left(u, 2, u^2\right) \quad \Rightarrow \quad \frac{d\overline{r}(u)}{du} = (1, 0, 2u)$$

$$P_1=(-1,2,1) \quad \Rightarrow \quad u_1 = -1$$

$$P_2=(1,2,1) \quad \Rightarrow \quad u_2 = +1$$

$$\int_L \overline{A} \cdot d\overline{r} = \int_{u_1}^{u_2} \overline{A}\left(\overline{r}(u)\right) \cdot \frac{d\overline{r}}{du}\, du$$

$$\overline{A} = \left(x^2, \quad \sinh(yz), \quad z^2\right) = \left(u^2, \quad \sinh(2u^2), \quad u^4\right)$$

$$\int_L \overline{A} \cdot d\overline{r} = \int_{-1}^{1} \left(u^2, \quad \sinh(2u^2), \quad u^4\right) \cdot (1, 0, 2u)\, du =$$

$$\int_{-1}^{1} \left(u^2 + 2u^5\right) du = \left[\frac{1}{3}u^3 + \frac{1}{3}u^6\right]_{-1}^{1} = \frac{2}{3}$$

PROBLEM 5

Calculate the flux of the vector field: $\overline{A} = (x^2 - y^2)\hat{e}_x + (x+y)^2\hat{e}_y + (x-y)^2\hat{e}_z$

through the surface S: $\overline{r}(u,v) = (u+v, u-v, uv)$

$$\begin{cases} u: & -1 \to 1 \\ v: & -1 \to 1 \end{cases}$$

$$\hat{n}\cdot\hat{e}_z > 0$$

SOLUTION

The flux is the integral: $\iint_S \overline{A}\cdot d\overline{S} = \int_v \int_u \overline{A}(\overline{r}(u,v))\cdot\left(\frac{\partial\overline{r}}{\partial u}\times\frac{\partial\overline{r}}{\partial v}\right)dudv$

- $$\overline{A}(\overline{r}(u,v)) = ((u+v)^2 - (u-v)^2, (u+v+u-v)^2, (u+v-u+v)^2) = (u^2+v^2+2uv-(u^2+v^2-2uv), 4u^2, 4v^2) = (4uv, 4u^2, 4v^2) = 4(uv, u^2, v^2)$$

- $$\left.\begin{aligned} \frac{d\overline{r}}{du} &= (1,1,v) \\ \frac{d\overline{r}}{dv} &= (1,-1,u) \end{aligned}\right\} \Rightarrow \frac{d\overline{r}}{du}\times\frac{d\overline{r}}{dv} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 1 & v \\ 1 & -1 & u \end{vmatrix} = (u+v, v-u, -2) \qquad \hat{n}\cdot\hat{e}_z > 0 \quad \Rightarrow \text{-S}$$

- $$\iint_{-S} \overline{A}\cdot d\overline{S} = 4\int_{-1}^{1}\int_{-1}^{1}(uv, u^2, v^2)\cdot(u+v, v-u, -2)dudv = 4\int_{-1}^{1}\int_{-1}^{1}(u^2v + uv^2 + u^2v - u^3 - 2v^2)dudv =$$

$$4\int_{-1}^{1}\left[\frac{u^3}{3}v + \frac{u^2}{2}v^2 + \frac{u^3}{3}v - \frac{u^4}{4} - 2uv^2\right]_{-1}^{1} dv = 4\int_{-1}^{1}\left(\frac{2v}{3} + \frac{2v}{3} - 4v^2\right)dv = 4\left[\frac{4v^2}{6} - \frac{4v^3}{3}\right]_{-1}^{1} = -\frac{32}{3}$$

$$\Rightarrow \iint_S \overline{A}\cdot d\overline{S} = -\iint_{-S} \overline{A}\cdot d\overline{S} = \frac{32}{3}$$

PROBLEM 6

Calculate the flux of the vector field: $\overline{A} = xy\hat{e}_x + yz\hat{e}_y + (xz+1)\hat{e}_z$

through the surface S: $\begin{cases} z = 4 - x^2 - y^2 \\ x^2 + y^2 \le 4 \\ \hat{n}\cdot\hat{e}_z \le 0 \quad \text{on S} \end{cases}$

SOLUTION

The flux is the integral: $\iint_S \overline{A}\cdot d\overline{S} = \int_v\int_u \overline{A}(\overline{r}(u,v))\cdot\left(\frac{\partial\overline{r}}{\partial u}\times\frac{\partial\overline{r}}{\partial v}\right)dudv$

Parameterization of S:

$$\left.\begin{aligned} x &= \rho\cos\theta \\ y &= \rho\sin\theta \\ z &= 4-\rho^2 \end{aligned}\right\} \Rightarrow \quad S: \quad \overline{r}(\rho,\theta) = \left(\rho\cos\theta, \rho\sin\theta, 4-\rho^2\right)$$

The normal is defined so that its z-componenet is negative.
Instead, with this parameterization we get a positive the z-component
→ we need to change sign to the final result

$$\left.\begin{aligned} \frac{d\overline{r}(\rho,\theta)}{d\rho} &= (\cos\theta, \sin\theta, -2\rho) \\ \frac{d\overline{r}(\rho,\theta)}{d\theta} &= (-\rho\sin\theta, \rho\cos\theta, 0) \end{aligned}\right\} \Rightarrow \left(\frac{d\overline{r}(\rho)}{d\rho}\times\frac{d\overline{r}(\rho,\theta)}{d\theta}\right) = \left(2\rho^2\cos\theta, 2\rho^2\sin\theta, \rho\right)$$

$$\iint_S \overline{A}\cdot d\overline{S} = \int_0^{2\pi}\int_0^2 \left(\rho^2\sin\theta\cos\theta, \left(4-\rho^2\right)\rho\sin\theta, \left(4-\rho^2\right)\rho\cos\theta + 1\right)\cdot\left(2\rho^2\cos\theta, 2\rho^2\sin\theta, \rho\right)d\rho d\theta =$$

$$= \int_0^{2\pi}\int_0^2 \left(2\rho^4\underbrace{\sin\theta\cos^2\theta} + 2\left(4-\rho^2\right)\rho^3\sin^2\theta + \left(4-\rho^2\right)\rho^2\underbrace{\cos\theta} + \rho\right)d\rho d\theta = \int_0^2\left(2\left(4-\rho^2\right)\rho^3\left[\left(\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right)\right]_0^{2\pi} + \rho\left[\theta\right]_0^{2\pi}\right)d\rho =$$

$\boxed{\textit{odd functions} \Rightarrow \textit{the integral in } \theta \textit{ over } 2\pi \textit{ is zero}}$

$$= \int_0^2\left(2\pi\left(4-\rho^2\right)\rho^3 + 2\pi\rho\right)d\rho = 2\pi\left[4\frac{\rho^4}{4} - \frac{\rho^6}{6}\right]_0^2 + \pi\left[\rho^2\right]_0^2 = 2\pi\left(16-\frac{32}{3}\right) + 4\pi = \frac{44}{3}\pi$$

$$\iint_S \overline{A}\cdot d\overline{S} = -\frac{44}{3}\pi$$