VEKTORANALYS

HT 2021

CELTE / CENMI

ED1110

GRADIENTEN

Kursvecka 1

Kapitel 4-5 (*Vektoranalys*, 1:e uppl, Frassinetti/Scheffel)

version: 29-aug-2021

Today:

- Scalar fields and vector fields
- Level surfaces and level curves
- The position vector
- The gradient
- The directional derivative
- Three theorems related to the gradient
- Scalar potential $\rightarrow$ home assignment (+övning week 2)

Connections with next topics and next weeks

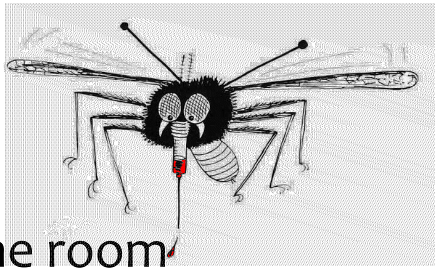
Scalar and vector fields:

- They will be used throughout the course
- They are essential for line integrals, flux, Gauss' theorem, Stokes' theorem…

Gradient:

- A wide range of applications in physics and engineering
- It has links with most of the course. For example
  - week 3 (Gauss' theorem),
  - week 4 (curvilenar coordinates)
  - week 5 (nabla)
  - week 6 (Laplace and Poisson's equations)

TARGET PROBLEM



A mosquito is flying in the room.
How does the mosquito find us in the dark?
A theory is: the mosquito flies toward the warmest region of the room
→ The mosquito must know:

(A) How the temperature $T(x,y,z)$ changes along the flying direction
(B) In which direction it must fly to be in a warmer place as quick as possible

Problem

The temperature is described by the scalar field: $T(x,y,z)=x^2+2yz-z$ [°C]
The mosquito is in the point P: (1,1,2)

- Question A: in which direction the mosquito must fly to be in a warmer place as quick as possible?
- Question B: How much does the temperature change in time if the mosquito flies with velocity 3m/s in direction $-2\hat{e}_x+2\hat{e}_y+\hat{e}_z$?

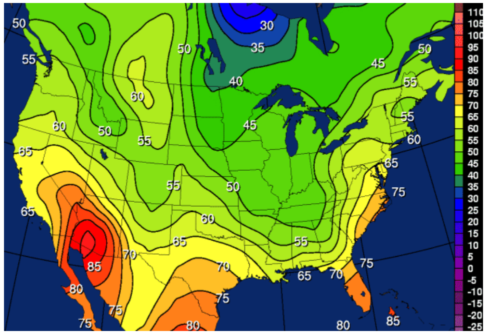
To solve the problem, we need to:

(1) introduce a **SCALAR FIELD**, $T(x,y,z)$
(2) measure the **rate of change of the scalar field** $T(x,y,z)$ in R^3
(3) find the **direction** along which **the rate of change** of $T(x,y,z)$ **is maximum**

SCALAR FIELD AND VECTOR FIELD

A scalar quantity is said to be a **field** if it is a function of position

A **scalar field** associates a real number $\phi(x,y,z)$ to each point (x,y,z) of the space.

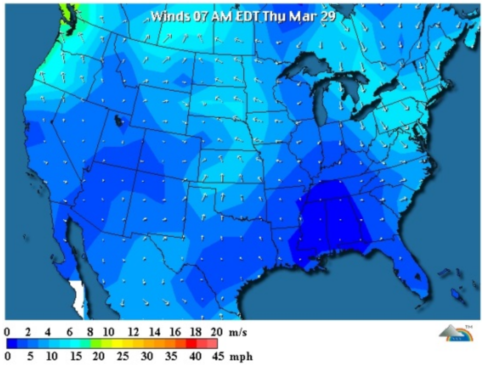


Examples:
- temperature distribution in the space
- pressure distribution in a fluid
- electrostatic potential around an electric charge

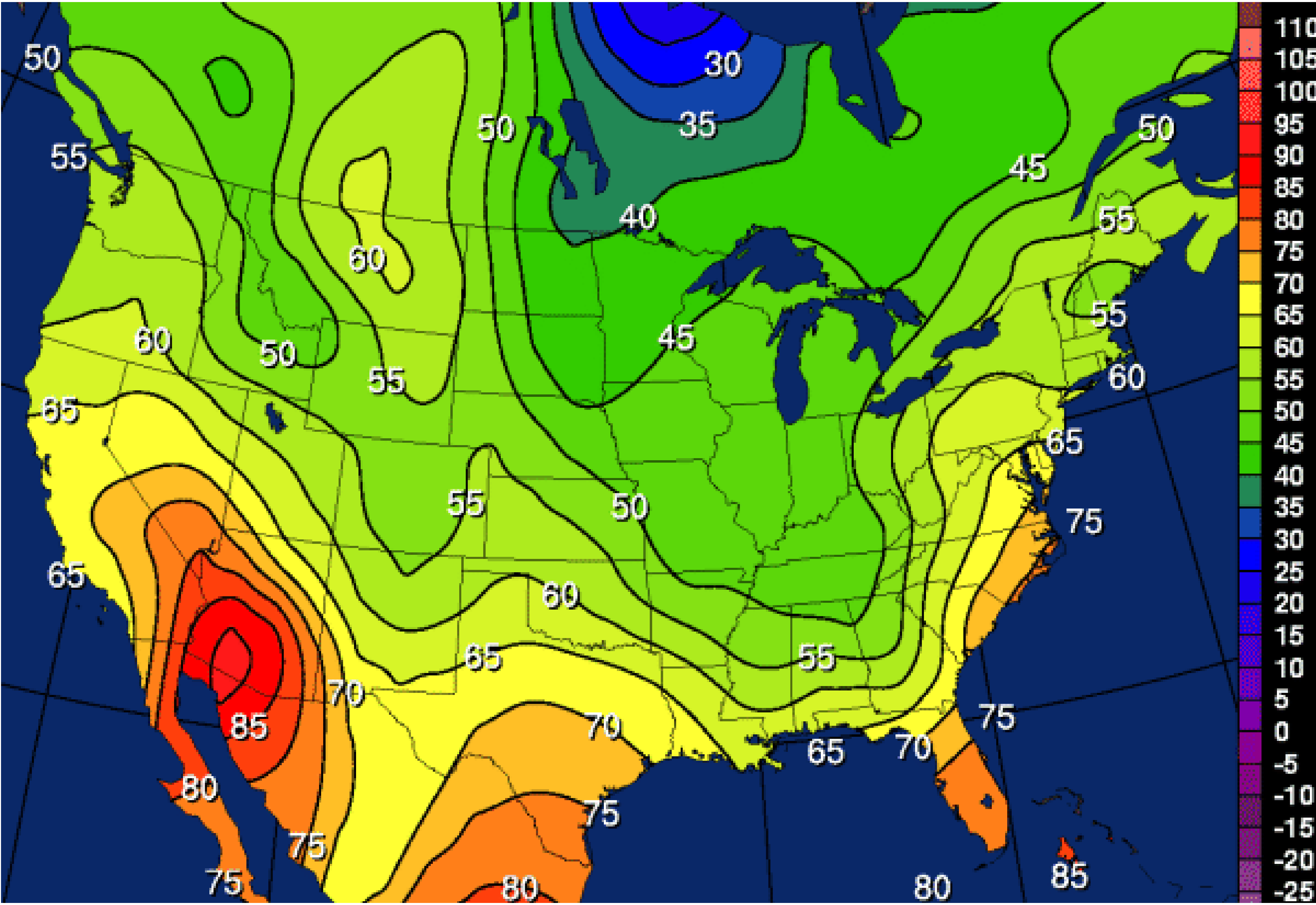
A vector quantity is said to be a **field** if it is a function of position

A **vector field** associates a vector $\overline{A}(x,y,z)$ to each point (x,y,z) of the space.

Examples:
- velocity distribution in a fluid
- magnetic field around a magnet
- electrostatic field around an electric charge



To solve our target problem, today we will focus on scalar fields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LEVEL CURVES and LEVEL SURFACES

- Level curves and level surfaces are useful to visualize a scalar field (in R^2 and R^3 respectively).
- What is a level curve?

  A curve on which the scalar field $\phi(x,y)$ is constant:

$$\phi(x,y)=c$$

- What is a level surface?

  A surface on which the scalar field $\phi(x,y,z)$ is constant:

$$\phi(x,y,z)=c \qquad (1)$$

- To create an "image" of the scalar field $\phi(x,y,z)$ we can consider a family of level surfaces:

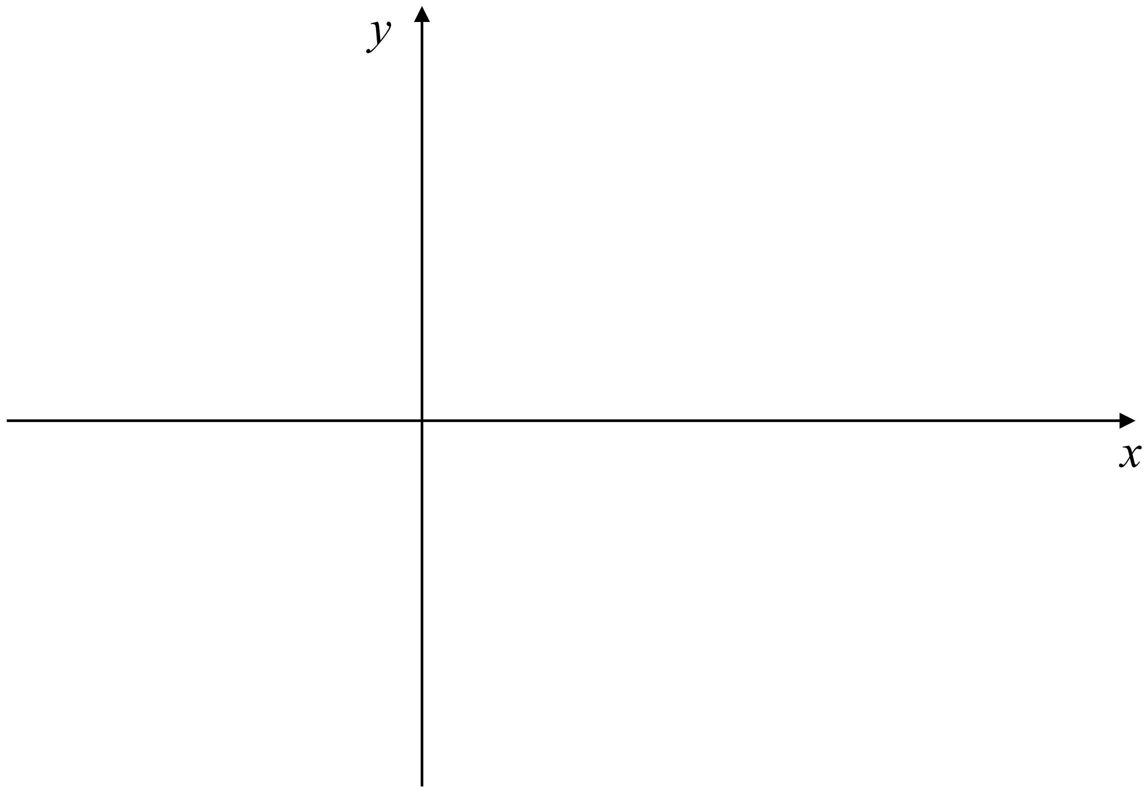
$$\phi(x,y,z)=c+nh \qquad \text{where h is a constant and } n=0, \pm1, \pm2, \pm3, \ldots \qquad (2)$$

Example: $\phi(x,y,z)=1 \quad \phi(x,y,z)=3 \quad \phi(x,y,z)=5 \quad \phi(x,y,z)=7 \;\ldots$

To improve the details of the "image", you can decrease the steps between one surface and the next.

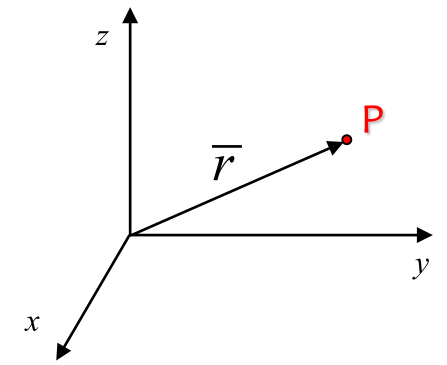
EXAMPLE

Plot the level curves of the scalar field: $\phi = \dfrac{x^2}{4} + y^2$



POSITION VECTOR

- The vector from the origin to the point P=(x,y,z) is called position vector $\overline{r}$
- Note that $\overline{r}$ depends on the choice of the coordinate system



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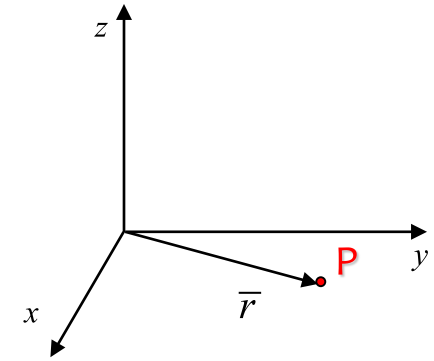

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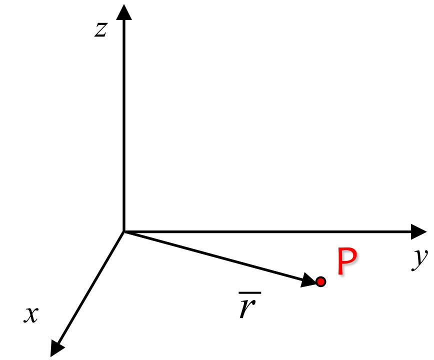


- The vector from the origin to the point P=(x,y,z) is called position vector $\overline{r}$
- Note that $\overline{r}$ depends on the choice of the coordinate system
- $\overline{r}$ can be expressed with different notations:

$$\overline{r} = \overline{r}(x, y, z) \qquad \overline{r} = (x, y, z) \qquad \overline{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$$

(in a Cartesian coordinate system)

- The absolute value of a position vector is the scalar:

$$r = |\overline{r}| = \sqrt{x^2 + y^2 + z^2} \tag{3}$$

- The differential of a position vector can be written as a vector whose components are the differential of each position vector component:

$$\overline{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$$

$$\Downarrow$$

$$d\overline{r} = dx\hat{e}_x + dy\hat{e}_y + dz\hat{e}_z \tag{4}$$

THE ELECTRIC FIELD GENERATED BY A POINT CHARGE
(a useful application of the position vector)

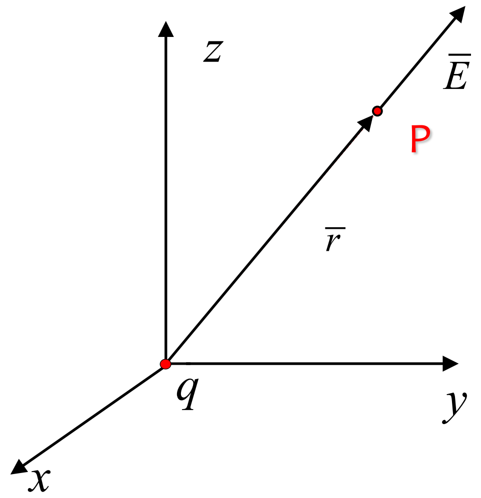
Consider a point charge q located in the origin and a point P defined by the position vector $\overline{r}$

- The absolute value of the of the electric field in P is:

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\overline{r}|^2}$$

- In this case, the direction of the electric field is from the origin to P, so it is given by: $\overline{r}$

$$\overline{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\overline{r}|^2}\frac{\overline{r}}{|\overline{r}|} = \frac{q}{4\pi\varepsilon_0}\frac{\overline{r}}{|\overline{r}|^3}$$



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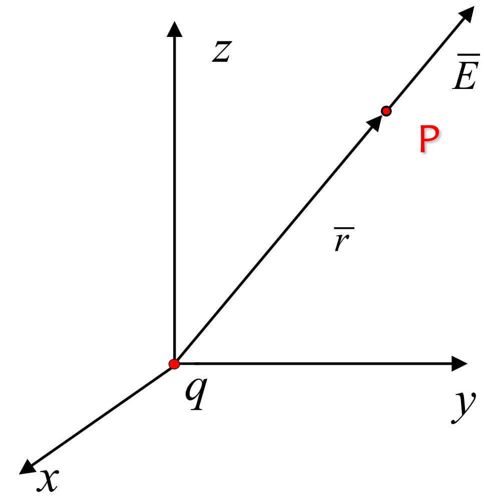

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- In this case, the direction of the electric field is from the origin to P, so it is given by: $\overline{r}$

$$\overline{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{\left|\overline{r}\right|^2}\frac{\overline{r}}{\left|\overline{r}\right|} = \frac{q}{4\pi\varepsilon_0}\frac{\overline{r}}{\left|\overline{r}\right|^3}$$

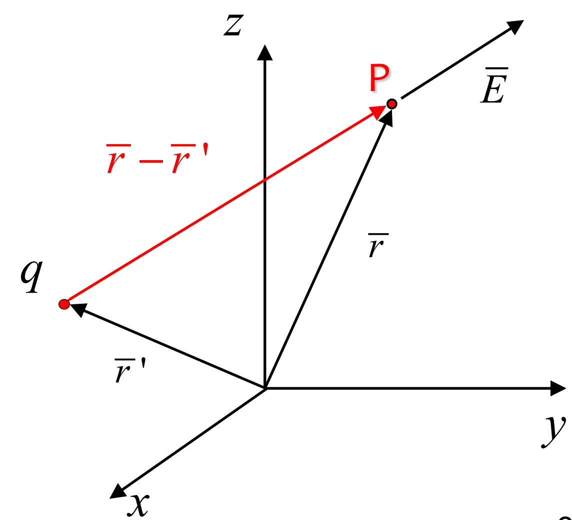
- How can we express the electric field if the charge is not in the origin?
  - $\overline{r}\,'$ is used to identify the position of the “source”
  - $\overline{r}$ is used to identify the position of P (where you need to calculate the field)

We need to identify the distance and the direction from q to P

direction: $\overline{r} - \overline{r}\,'$

distance: $\left|\overline{r} - \overline{r}\,'\right|$

$$\overline{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{\left|\overline{r} - \overline{r}\,'\right|^2}\frac{\overline{r} - \overline{r}\,'}{\left|\overline{r} - \overline{r}\,'\right|} = \boxed{\frac{q}{4\pi\varepsilon_0}\frac{\overline{r} - \overline{r}\,'}{\left|\overline{r} - \overline{r}\,'\right|^3}}$$



THE GRADIENT

- Assume that $\phi(\bar{r})$ is a continuous and derivable scalar field
- DEFINITION:

in a Cartesian coordinate system

$$\boxed{grad\,\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = \frac{\partial\phi}{\partial x}\hat{e}_x + \frac{\partial\phi}{\partial y}\hat{e}_y + \frac{\partial\phi}{\partial z}\hat{e}_z} \quad (5)$$

IMPORTANT: the gradient of a scalar field is a vector field

EXERCISE: calculate the gradient of the vector field: $\phi = \frac{x^2}{4} + y^2$ and plot $grad\phi$ in the point *P: (2,0)* and in the point *Q: (0,-1)*

- Scalar field differential:

$$d\phi = grad\,\phi \cdot d\bar{r} \quad (6)$$

- Let's introduce :
  - the amplitude of the position vector differential, *d*s, and
  - the direction ê (ê is a *unit vector, i.e.* |ê|=*1*)

$$d\bar{r} = \hat{e}\,ds \quad (7)$$

Equations (6) and (7) give:

Directional derivative

$$\boxed{\frac{d\phi}{ds} = grad\,\phi \cdot \hat{e}} \quad (8)$$

The rate of variation of ϕ in a given direction corresponds to the component of the vector gradient in that direction

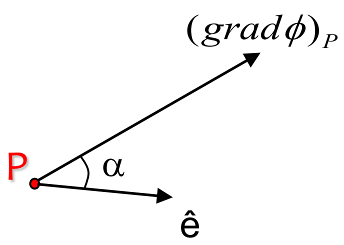
Answer to question (A)

THE GRADIENT

THEOREM 1 *(4.1 in the textbook)*

The direction of the maximum growth (rate of change) of a scalar field ϕ in the point P is the direction of the gradient in the point P.
The maximum growth (rate of change) of ϕ per unit length is $|(grad\,\phi)_P|$

PROOF



a- let's calculate the derivative in the direction ê *Eq. (8)*

$$\frac{d\phi}{ds} = grad\,\phi \cdot \hat{e} = |grad\,\phi| \cos\alpha$$

b- this is maximum when:

$\cos\alpha = 1$

which implies:

$\alpha = 0$ (ê // $grad\,\phi$) and $\dfrac{d\phi}{ds} = |grad\,\phi|$

Answer to question (B)

THE GRADIENT

THEOREM 2

If ϕ has a maximum or a minimum in the point P, then the gradient in P is zero.

PROOF

From Equation (8): $$\frac{d\phi}{ds} = grad\phi \cdot \hat{e}$$

ϕ has a maximum or a minimum in P $\Rightarrow$ $d\phi/ds=0$

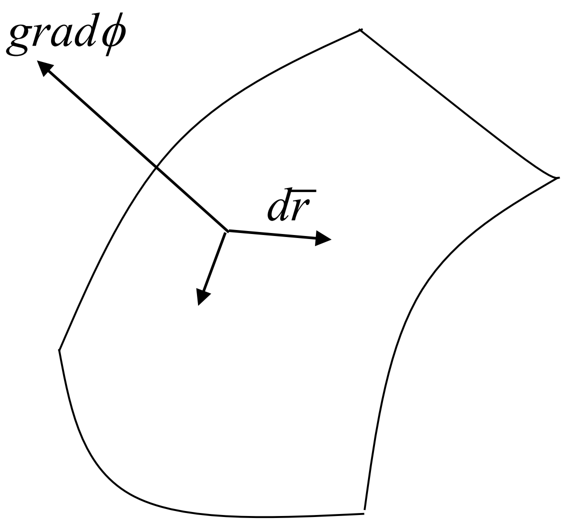
using equation (8), $d\phi/ds=0$ implies gradϕ=0

THE GRADIENT

THEOREM 3 *(4.2 in the textbook)*

The gradient of a scalar field $\phi(x,y,z)$ in the point P is orthogonal to the level surface ϕ=c in P.

PROOF



a- Let's do a small movement $d\bar{r}$ along the level surface

b- Remember that on the level surface ϕ is constant:

$$d\phi=0$$

c- Then, using equation (6):

$$d\phi = grad\phi \cdot d\bar{r} = 0$$

d- This implies that $grad\phi$ is perpendicular to $d\bar{r}$

e- $grad\phi$ is perpendicular to each $d\bar{r}$ on the level surface $grad\phi$ is perpendicular to the level surface

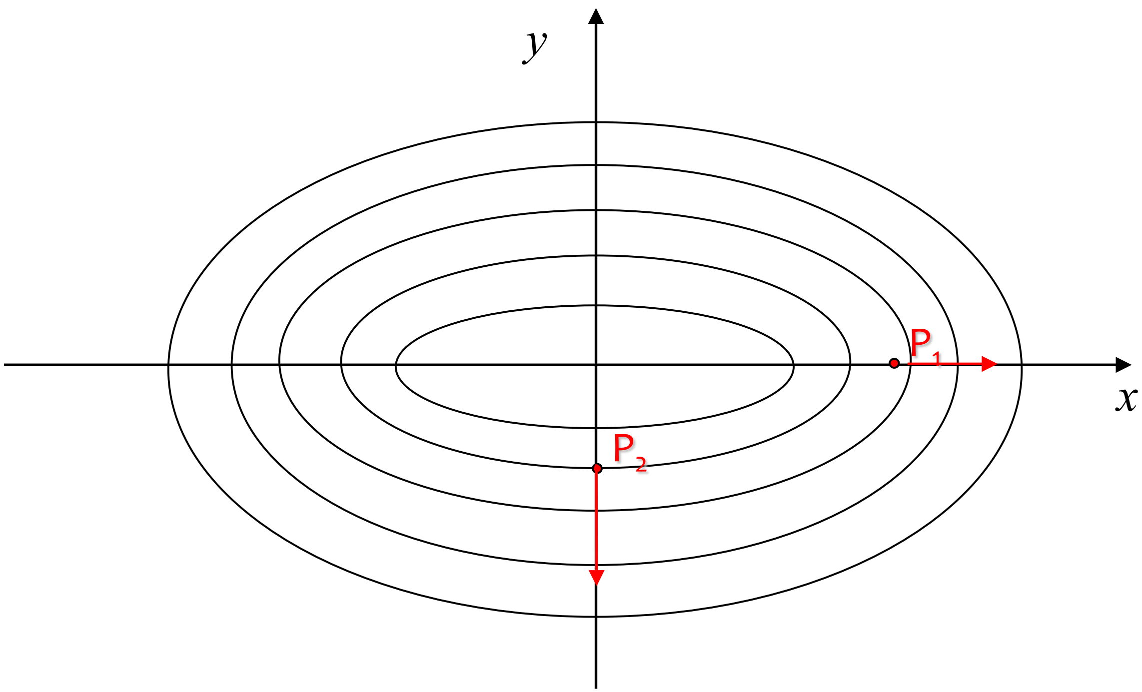
2D-EXAMPLE

- Theorems 1, 2 and 3 are valid also in two dimensions.
- $grad\phi$ is a vector field that:
  - in each point is orthogonal to the level curve in that point and
  - always points along the direction in which the height grows faster

Plot the level curves of the scalar field: $\phi = \dfrac{x^2}{4} + y^2$

and calculate the gradient in the points $P_1=(2,0)$ and $P_2=(0,-1)$

$$grad\phi = \left(\frac{x}{2}, 2y\right)$$



$$grad\phi\big|_{P1} = (1,0)$$

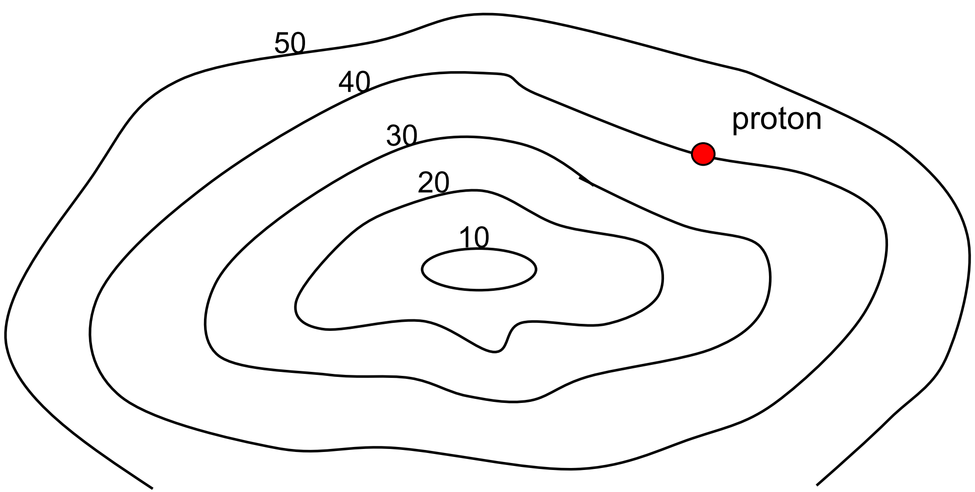
$$grad\phi\big|_{P2} = (0,-2)$$

ELECTROSTATIC POTENTIAL AND ELECTRIC FIELD

- Consider an electrostatic potential $V(\overline{r})$
- The electric field produced by $V(\overline{r})$ is: $\overline{E}(\overline{r}) = -grad\, V(\overline{r})$

  (see the TET course for details)
- The force produced by $\overline{E}(\overline{r})$ on an electric charge q is: $\overline{F}(\overline{r}) = q\overline{E} = -q\, grad\, V(\overline{r})$

EXERCISE:
- Consider a proton in an electrostatic potential:



In which direction will the proton move?

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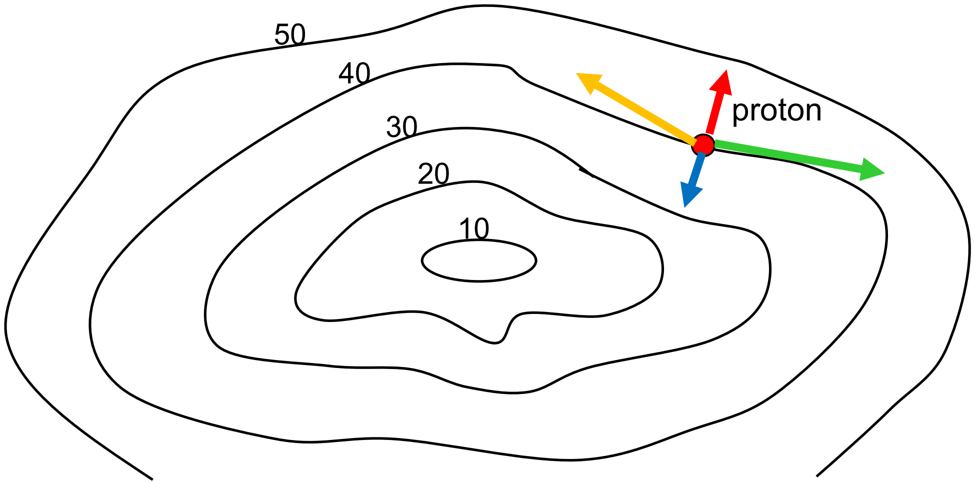


In which direction will the proton move?

Poll 6

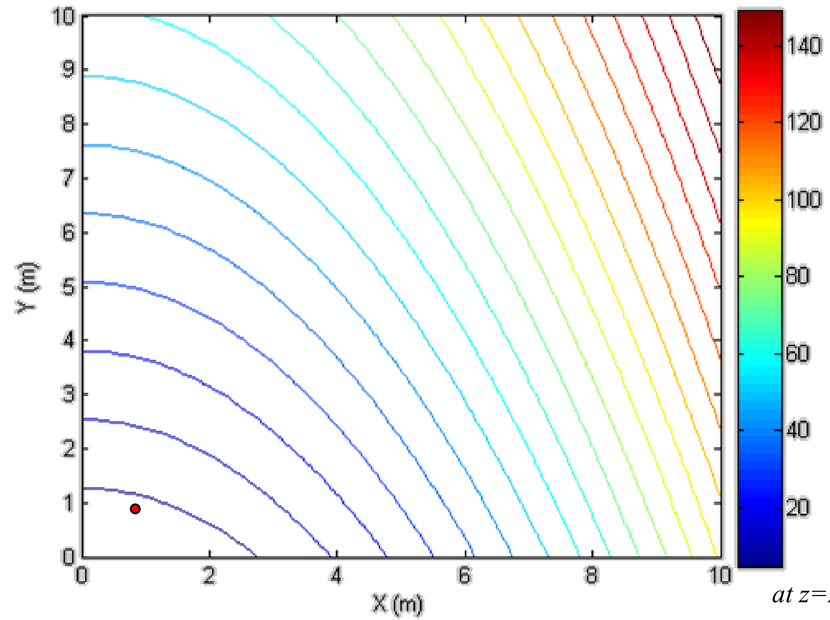
TARGET PROBLEM

A mosquito is flying around in the room.

The temperature is described by the scalar field:

$$T(x,y,z)=x^2+2yz-z \quad [^\circ C]$$

The mosquito is in the point P=(1,1,2)



(a) In which direction the mosquito will fly to be in a warmer place as quick as possible?

(b) How much the temperature changes in time if the mosquito flies with velocity 3m/s in direction (-2,2,1)?

TARGET PROBLEM

(a) In which direction the mosquito will fly to be warm as quick as possible?

We use theorem 1: The gradient in the point P is a vector that points to the direction in which the scalar field in P has the highest growth.

From definition (5):

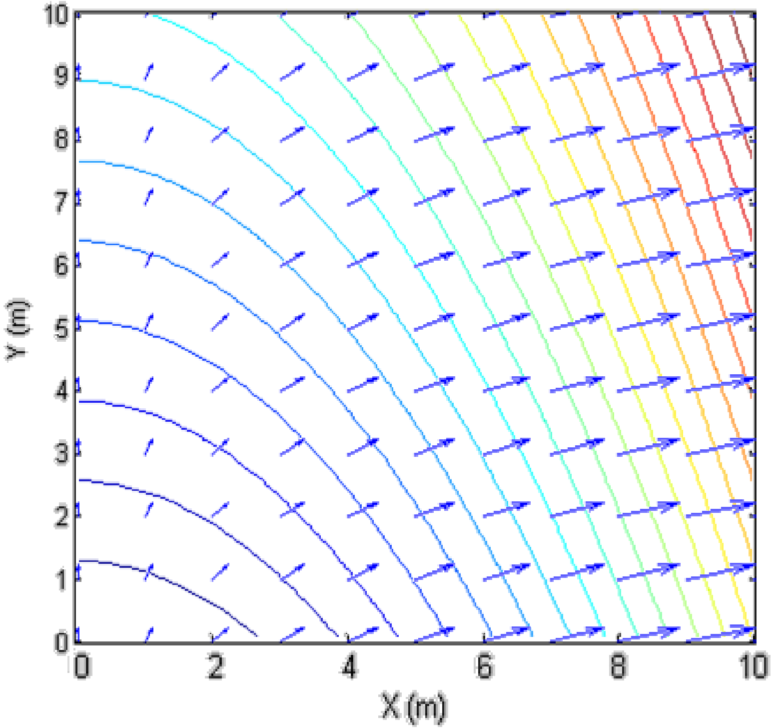
$$gradT = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right)$$

$$T(x, y, z) = x^2 + 2yz - z$$

$$\frac{\partial T}{\partial x} = 2x, \qquad \frac{\partial T}{\partial y} = 2z, \qquad \frac{\partial T}{\partial z} = 2y - 1$$

$$gradT = (2x, 2z, 2y - 1)$$

at z=2



The mosquito is in P:(1,1,2)

$$(gradT)_{P=(1,1,2)} = (2 \cdot 1, 2 \cdot 2, 2 \cdot 1 - 1) = (2, 4, 1)$$

The mosquito will fly in direction (2,4,1)

TARGET PROBLEM

(b) How fast the temperature changes in time if the mosquito flies with velocity 3m/s in direction (-2,2,1)?

We must calculate $\dfrac{dT}{dt}$ where t is the time

Using equation (6):

$$\frac{dT}{dt} = gradT \cdot \frac{d\bar{r}}{dt} = gradT \cdot \hat{e}\frac{ds}{dt}$$

where
$$\left.\begin{cases} \dfrac{ds}{dt} = \left|\bar{v}\right| = 3m/s \\ \hat{e} = \dfrac{\bar{v}}{\left|\bar{v}\right|} = \dfrac{(-2,2,1)}{\left|(-2,2,1)\right|} = \dfrac{(-2,2,1)}{\sqrt{(-2)^2+2^2+1^2}} = \dfrac{(-2,2,1)}{3} \end{cases}\right\} \Longrightarrow \frac{d\bar{r}}{dt} = \frac{(-2,2,1)}{3}\cdot 3 = (-2,2,1)$$

$$\frac{dT}{dt} = gradT \cdot \frac{d\bar{r}}{dt} = (2,4,1)\cdot(-2,2,1) = 5 \quad [\text{C/s}]$$

WHICH STATEMENT IS WRONG?

1- A scalar field associates a real number to a point in space

2- The increase of a scalar field in a given direction can be calculated with the directional derivative: $\frac{d\phi}{ds} = grad\phi \cdot \hat{e}$

3- If ϕ is a scalar field, then

$$grad\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)$$ in a spherical coordinate system

4- A vector field can be written as $\overline{A} = \overline{A}(x, y, z)$

Poll 1