VEKTORANALYS /ED1110

HT 2021

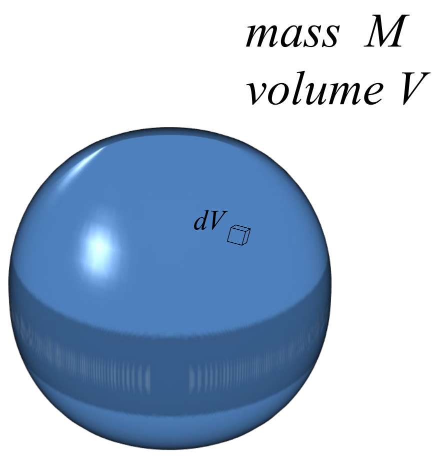
CELTE / CENMI

PRACTICAL EXAMPLES USEFUL FOR VECTOR ANALYSIS

version: 21-aug-2021

VOLUMETRIC DENSITY

Probably, you are familiar with the mass density:



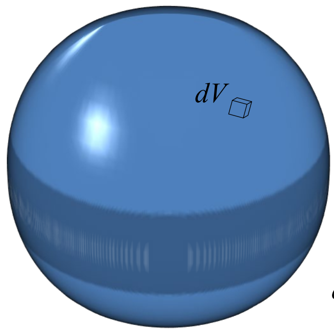
- The volumetric mass density of a substance is its mass per unit volume.
- If the substance is uniform, the volumetric mass density is defined as: $\rho = M/V$
  where M is the mass and V the volume
- Example: the mass density of the water is ρ=1000kg/m^3.
- If the material is not uniform, the mass density is not constant.
  In this case, it is defined as: $\rho(x, y, z) = dM \,/\, dV$
- If you know the volumetric mass density of a body, you can calculate its mass with a volume integral: $M = \int_V \rho dV$

In analogy with the mass, it is possible to define the density of other quantities, for example, the electric charge.

ELECTRIC CHARGE DENSITY

The **volumetric charge density** it is the electric charge per unit volume:

total electric charge Q
volume V



$$\rho_V = \frac{dQ}{dV}$$

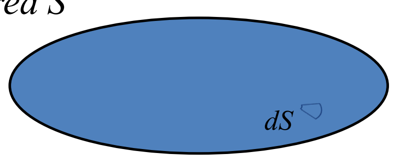
$$dQ = \rho_V dV$$
$$Q = \int dQ$$

$$Q = \int_V \rho_V dV$$

dV is an infinitesimal volume element in the volume V

The **surface charge density** is the electric charge per unit area:

total electric charge Q
area S



$$\rho_S = \frac{dQ}{dS}$$

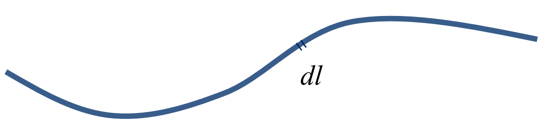
$$dQ = \rho_S dV$$
$$Q = \int dQ$$

$$Q = \int_S \rho_S dS$$

dS is an infinitesimal surface element on the surface S

The **linear charge density** is the electric charge per unit length:

total electric charge Q
curve L



$$\rho_l = \frac{dQ}{dl}$$

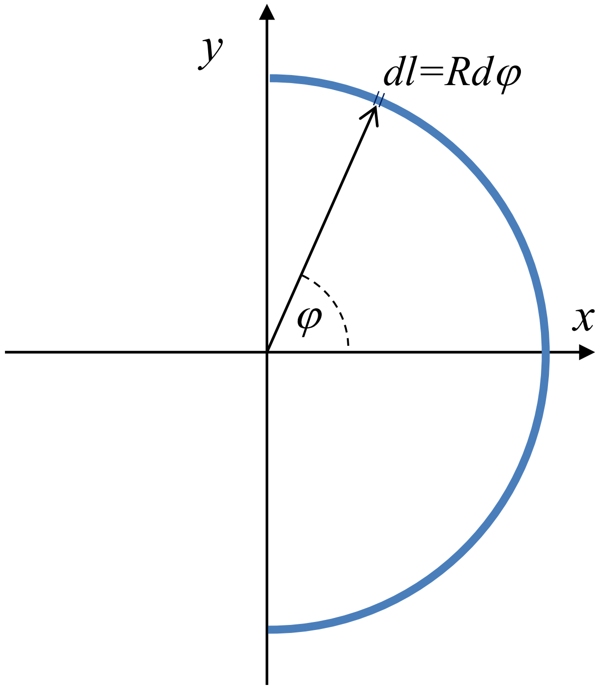
$$dQ = \rho_l dV$$
$$Q = \int dQ$$

$$Q = \int_L \rho_l dl$$

dl is an infinitesimal line element along the curve L

LINEAR ELECTRIC CHARGE DENSITY: EXAMPLE

Consider a circular arc with radius R, centered in the origin and from $\varphi=-\pi/2$ to $\varphi=+\pi/2$



EXAMPLE 1:

The arc is electrically charged. The line charge density in the arc is not uniform:

$$\rho_l = \rho_0 \cos\varphi$$

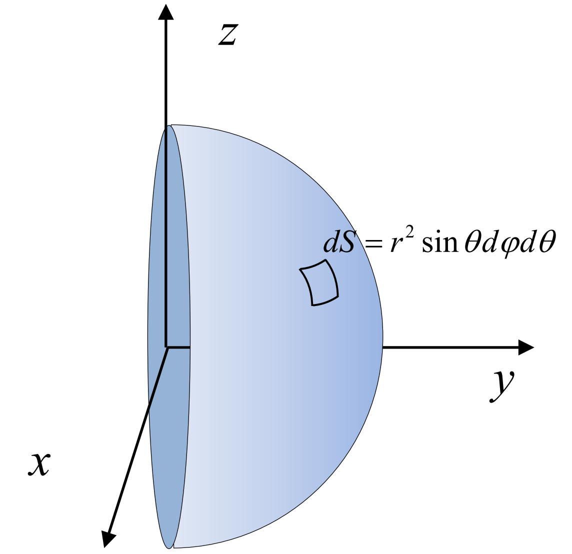
where ρ_0 is constant.
Calculate the total electric charge in the arc.

$$\left.\begin{aligned} dq &= \rho_l dl \\ dl &= R d\varphi \end{aligned}\right\} \Rightarrow dq = \rho_0 \cos\varphi R d\varphi$$

$$Q = \int dq = \int_{-\pi/2}^{\pi/2} \rho_0 \cos\varphi R d\varphi = \rho_0 R \left[\sin\varphi\right]_{-\pi/2}^{\pi/2} = 2\rho_0 R$$

SURFACE ELECTRIC CHARGE DENSITY: EXAMPLE

Consider a half spherical shell with radius R and centered in the origin and with base parallel to the x-z plane



EXAMPLE 2:

The shell is electrically charged. The surface charge density in the shell is not uniform:

$$\rho_S = \rho_0 \sin\varphi$$

where ρ_0 is constant.

Calculate the total electric charge in the shell.

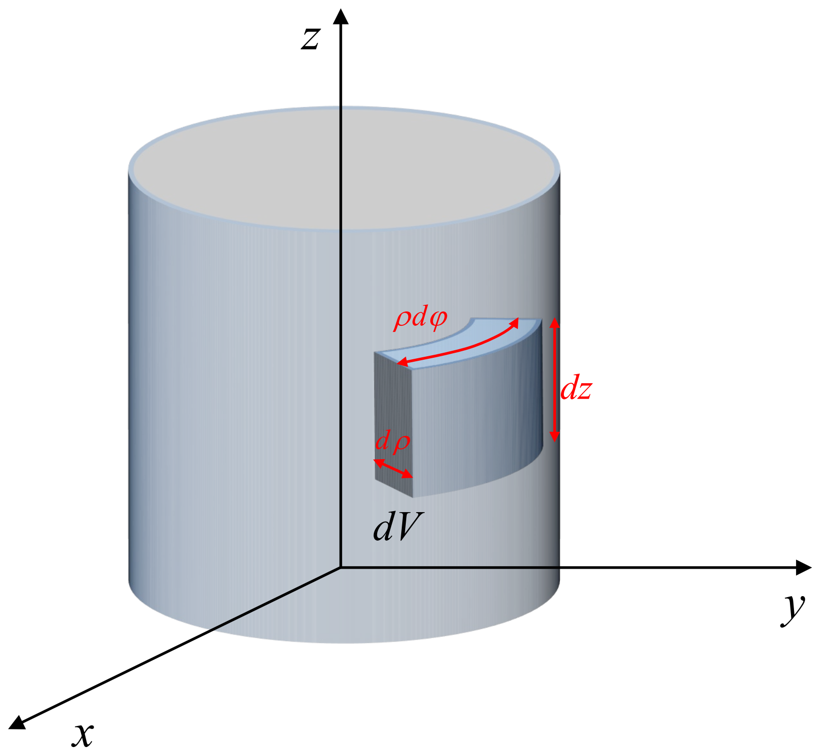
$$\left.\begin{array}{l} dq = \rho_S dS \\ dS = r^2 \sin\theta d\varphi d\theta \end{array}\right\} \Rightarrow dq = \rho_0 \sin\varphi r^2 \sin\theta d\varphi d\theta$$

The radius is R (and it is constant) $\Rightarrow dq = \rho_0 R^2 \sin\varphi \sin\theta d\varphi d\theta$

$$Q = \int dq = \int_0^\pi \int_0^\pi \rho_0 R^2 \sin\varphi d\varphi \sin\theta d\theta = \rho_0 R^2 \left[-\cos\varphi\right]_0^\pi \left[-\cos\theta\right]_0^\pi = 4\rho_0 R^2$$

VOLUMETRIC ELECTRIC CHARGE DENSITY: EXAMPLE

Consider a cylinder with radius R, height L, with axis along the z-axis and base on the x-y plane



EXAMPLE 3:

The volume charge density in the cylinder is uniform: $\rho_V=\rho_0$
where ρ_0 is constant.
Calculate the total charge in the cylinder.

$$dq = \rho_V dV$$

$$dV = \rho d\varphi d\rho dz$$

$$Q = \int dq = \int_0 \rho_V \rho d\varphi d\rho dz = \int_0^L \int_0^R \int_0^{2\pi} \rho_V \rho d\varphi d\rho dz = \pi \rho_0 R^2 L$$

EXAMPLE 4:

The volume charge density in the cylinder is not uniform: $\rho_V=\rho_0(1-\rho/R)$
where ρ_0 is constant.
Calculate the total charge in the cylinder.

$$\left.\begin{aligned} dq &= \rho_V dV \\ dV &= \rho d\varphi d\rho dz \end{aligned}\right\} \Rightarrow dq = \rho_0\left(1-\frac{\rho}{R}\right)\rho d\varphi d\rho dz$$

$$Q = \int dq = \int_0^L \int_0^R \int_0^{2\pi} \rho_0\left(1-\frac{\rho}{R}\right)\rho d\varphi d\rho dz = 2\pi L\rho_0 \int_0^R \left(1-\frac{\rho}{R}\right)\rho d\rho = 2\pi L\rho_0 \int_0^R \left(\rho-\frac{\rho^2}{R}\right)d\rho =$$

$$= 2\pi L\rho_0\left[\frac{\rho^2}{2}-\frac{\rho^3}{3R}\right]_0^R = 2\pi L\rho_0 \frac{R^2}{6} = \frac{\pi}{3}\rho_0 L R^2$$

ELECTRIC FIELD GENERATED BY SEVERAL POINT CHARGES

The electric field in P generated by one point charge q is:

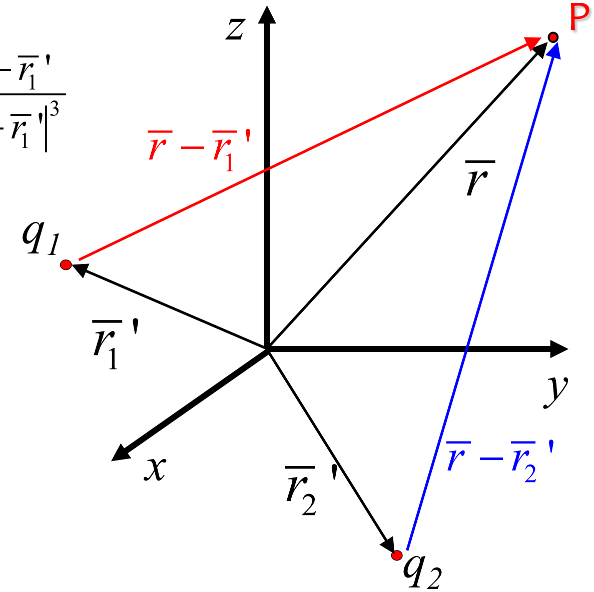
$$\bar{E} = \frac{q}{4\pi\varepsilon_0} \frac{\bar{r} - \bar{r}_1'}{\left|\bar{r} - \bar{r}_1'\right|^3}$$

The electric field generated by two point charges q_1 and q_2 is:

$$\bar{E} = \frac{q_1}{4\pi\varepsilon_0} \frac{\bar{r} - \bar{r}_1'}{\left|\bar{r} - \bar{r}_1'\right|^3} + \frac{q_2}{4\pi\varepsilon_0} \frac{\bar{r} - \bar{r}_2'}{\left|\bar{r} - \bar{r}_2'\right|^3}$$

The electric field generated by N point charges is:

$$\bar{E} = \sum_{i=1}^{N} \frac{q_i}{4\pi\varepsilon_0} \frac{\bar{r} - \bar{r}_i'}{\left|\bar{r} - \bar{r}_i'\right|^3}$$



ELECTRIC FIELD GENERATED BY A CHARGE DISTRIBUTION

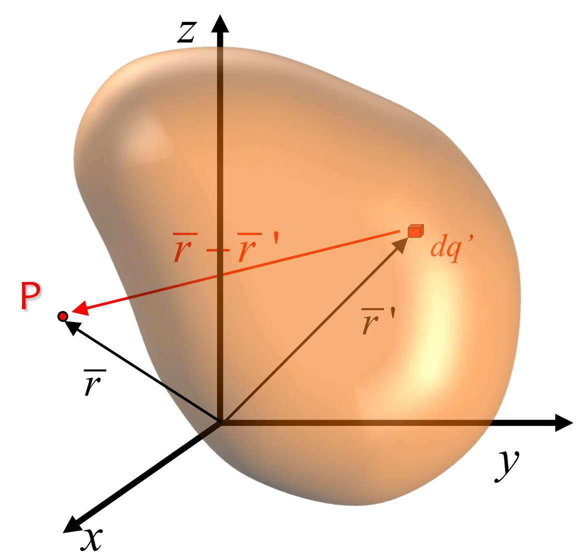
Let's consider an electrically charge object with a specific charge distribution. We want the total electric field in a point P.

The electric field produced in the point P by an infinitesimal charge is:

$$d\bar{E} = \frac{dq'}{4\pi\varepsilon_0} \frac{\bar{r} - \bar{r}'}{\left|\bar{r} - \bar{r}'\right|^3}$$

where $\bar{r}'$ is the position vector of dq' and $\bar{r}$ is the position vector of P.
Then the total electric field is the integral of $d\bar{E}$:

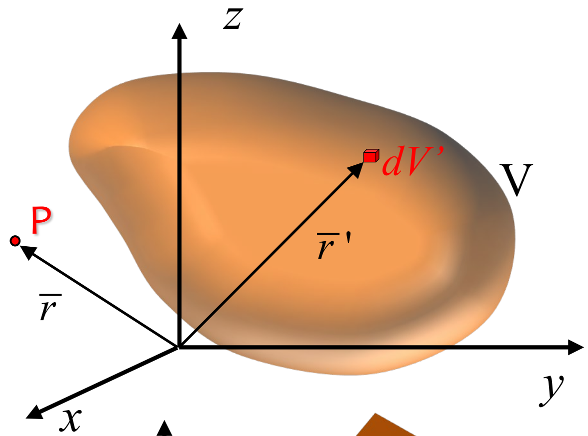
$$\bar{E} = \int \frac{dq'}{4\pi\varepsilon_0} \frac{\left(\bar{r} - \bar{r}'\right)}{\left|\bar{r} - \bar{r}'\right|^3}$$



ELECTRIC FIELD GENERATED BY A CHARGE DISTRIBUTION

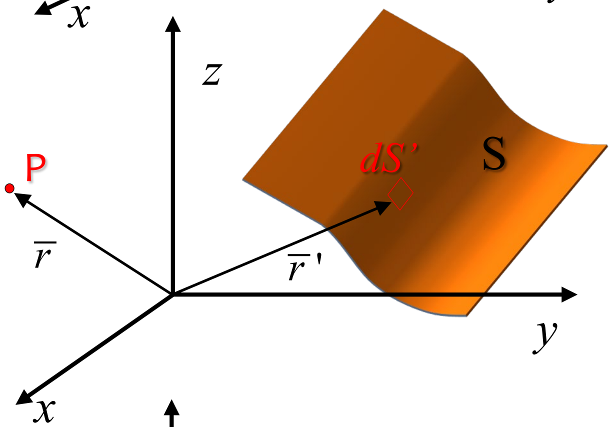
If the charge is spread in a **volume**: $dq = \rho_V dV$

$$\overline{E} = \int_V \frac{\rho_V}{4\pi\varepsilon_0} \frac{(\overline{r} - \overline{r}')}{|\overline{r} - \overline{r}'|^3} dV'$$



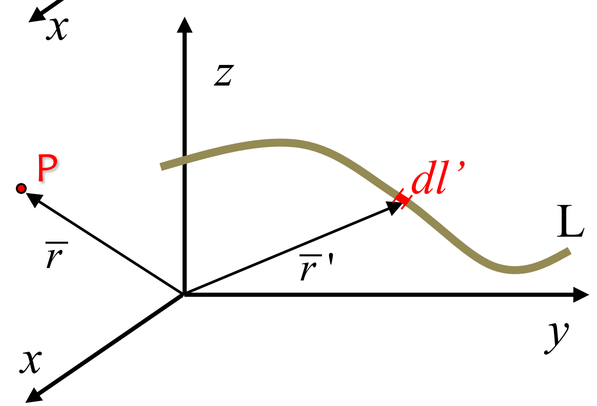
If the charge is spread on a **surface**: $dq = \rho_S dS$

$$\overline{E} = \int_S \frac{\rho_S}{4\pi\varepsilon_0} \frac{(\overline{r} - \overline{r}')}{|\overline{r} - \overline{r}'|^3} dS'$$



If the charge is spread on a **curve**: $dq = \rho_l dl$

$$\overline{E} = \int_L \frac{\rho_l}{4\pi\varepsilon_0} \frac{(\overline{r} - \overline{r}')}{|\overline{r} - \overline{r}'|^3} dl'$$



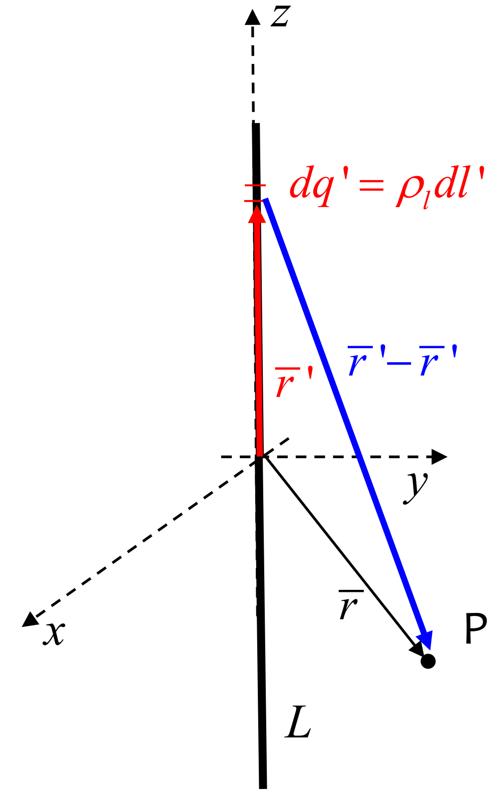
$\overline{r}$ is the position vector of the P (where we want to calculate the field)

$\overline{r}'$ is the position vector of the infinitesimal charge

EXAMPLE: the electric field generated by a straight wire

A straight wire "L" with charge density ρ_l (constant) has length $2L_0$, is located along the z-axis and centered at z=0.

Calculate the electric field on the plane z=0 produced by the wire L.



$$\overline{E}(\overline{r}) = \frac{1}{4\pi\varepsilon_0} \int_L \rho_l \frac{(\overline{r} - \overline{r}')}{|\overline{r} - \overline{r}'|^3} dl'$$

$$\overline{r}' = (0, 0, z') = z'\hat{e}_z$$

$$dl' = dz'$$

$$\overline{r} = \rho\hat{e}_\rho + z\hat{e}_z = \rho\hat{e}_\rho \quad \Rightarrow \quad \overline{r} - \overline{r}' = \rho\hat{e}_\rho - z'\hat{e}_z \quad \Rightarrow \quad |\overline{r} - \overline{r}'| = \sqrt{\rho^2 + z'^2}$$

$$\frac{(\overline{r} - \overline{r}')}{|\overline{r} - \overline{r}'|^3} dl' = \frac{\rho\hat{e}_\rho - z'\hat{e}_z}{\sqrt{\rho^2 + z'^2}^3} dz'$$

$$\overline{E}(\overline{r}) = \frac{\rho_l}{4\pi\varepsilon_0} \int_L \frac{\rho\hat{e}_\rho - z'\hat{e}_z}{\sqrt{\rho^2 + z'^2}^3} dz' = \frac{\rho_l}{4\pi\varepsilon_0} \int_L \frac{\rho\hat{e}_\rho}{\sqrt{\rho^2 + z'^2}^3} dz' - \frac{\rho_l}{4\pi\varepsilon_0} \int_L \frac{z'\hat{e}_z}{\sqrt{\rho^2 + z'^2}^3} dz' =$$

$$= \frac{\rho_l}{4\pi\varepsilon_0} \rho\hat{e}_\rho \left[\frac{z'}{\rho^2\sqrt{\rho^2 + z'^2}} \right]_{-L_0}^{L_0} - \frac{\rho_l}{4\pi\varepsilon_0} \hat{e}_z \left[-\frac{1}{\sqrt{\rho^2 + z'^2}} \right]_{-L_0}^{L_0} = \frac{\rho_l}{4\pi\varepsilon_0} \frac{2L_0}{\rho\sqrt{\rho^2 + L_0^2}} \hat{e}_\rho$$

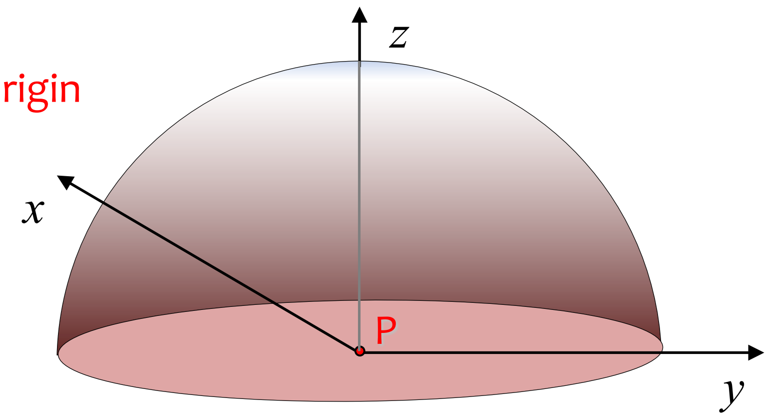
$\hat{e}_\rho$ does not depend on z, so you can move it out from the integration. But, in general, $\hat{e}_\rho$ is not constant so you need to integrate it.

$\hat{e}_z$ is constant, so you can always move it out from the integration.

EXAMPLE: the electric field generated by half sphere

A half sphere with radius R is centered in the origin and has the base on the x-y plane. The volume charge density in the sphere is: $\rho_V(r)=\rho_0\left(1-\frac{r}{R}\right)$

Calculate the electric field in the point P located in the origin



$$\bar{E}=\int_V \frac{\rho_V}{4\pi\varepsilon_0}\frac{(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3}dV'$$

$$\left.\begin{array}{l}\bar{r}=(0,0,0)=\bar{0}\\ \bar{r}'=r'\hat{e}_r\end{array}\right\}\Rightarrow\left\{\begin{array}{l}\bar{r}-\bar{r}'=-r'\hat{e}_r\\ |\bar{r}-\bar{r}'|=r'\end{array}\right.$$

$$dV'=r'^2\sin\theta' d\varphi' d\theta' dr'$$

$$\bar{E}=-\frac{\rho_0}{4\pi\varepsilon_0}\int_V\left(1-\frac{r'}{R}\right)\frac{r'\hat{e}_r}{r'^3}r'^2\sin\theta' d\varphi' d\theta' dr'=-\frac{\rho_0}{4\pi\varepsilon_0}\int_0^R\int_0^{\pi/2}\int_0^{2\pi}\left(1-\frac{r'}{R}\right)\hat{e}_r\sin\theta' d\varphi' d\theta' dr'$$

$$=-\frac{\rho_0}{4\pi\varepsilon_0}\int_0^R\int_0^{\pi/2}\int_0^{2\pi}\left(1-\frac{r'}{R}\right)\left(\sin\theta'\cos\varphi'\hat{e}_x+\sin\theta'\sin\varphi'\hat{e}_y+\cos\theta'\hat{e}_z\right)\sin\theta' d\varphi' d\theta' dr'$$

$$=-\frac{\rho_0}{4\pi\varepsilon_0}\int_0^R\int_0^{\pi/2}\int_0^{2\pi}\left(1-\frac{r'}{R}\right)\left(\sin^2\theta'\cos\varphi'\hat{e}_x\right)d\varphi' d\theta' dr'-\frac{\rho_0}{4\pi\varepsilon_0}\int_0^R\int_0^{\pi/2}\int_0^{2\pi}\left(1-\frac{r'}{R}\right)\left(\sin^2\theta'\sin\varphi'\hat{e}_y\right)d\varphi' d\theta' dr'$$

$$-\frac{\rho_0}{4\pi\varepsilon_0}\int_0^R\int_0^{\pi/2}\int_0^{2\pi}\left(1-\frac{r'}{R}\right)\left(\sin\theta'\cos\theta'\hat{e}_z\right)d\varphi' d\theta' dr'=$$

$$=-\frac{\rho_0}{4\pi\varepsilon_0}2\pi\hat{e}_z\int_0^{\pi/2}(\sin\theta'\cos\theta')d\theta'\int_0^R\left(1-\frac{r'}{R}\right)dr'=-\frac{\rho_0}{2\varepsilon_0}\hat{e}_z\left[-\frac{\cos^2\theta'}{2}\right]_0^{\pi/2}\left[r'-\frac{r'^2}{2R}\right]_0^R=-\frac{\rho_0}{2\varepsilon_0}\hat{e}_z\frac{1}{2}\frac{R}{2}=-\frac{\rho_0 R}{8\varepsilon_0}\hat{e}_z$$

EXAMPLE: electrostatic potential generated by a straight wire (using cartesian coordinates)

The electrostatic potential ϕ in the point P (x_p, y_p, z_p) produced by an electrically charged wire "C" with constant charge density λ is:

$$\phi(\bar{r}) = \int_C \frac{\lambda}{4\pi\varepsilon_0} \frac{|d\bar{r}\,'|}{|\bar{r} - \bar{r}\,'|}$$

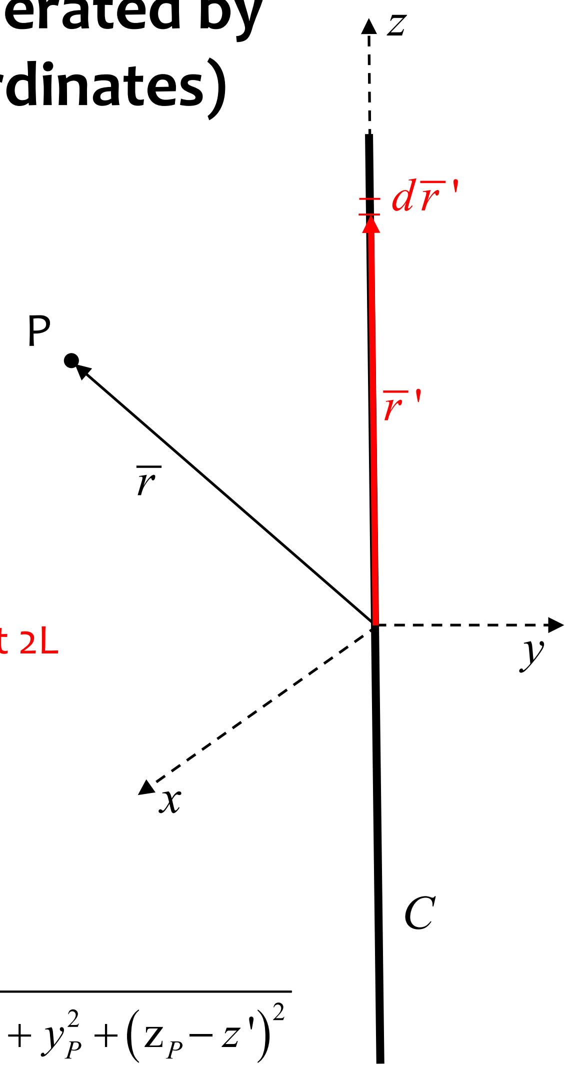
where:

$\bar{r}$ is the position vector of the point

$d\bar{r}\,'$ is an infinitesimal vector on the curve C

$\bar{r}\,'$ is the vector from the origin to $d\bar{r}\,'$

Exercise: calculate the potential produced by a straight wire of lenght 2L



Step 1. Calculate the term $\frac{|d\bar{r}\,'|}{|\bar{r} - \bar{r}\,'|}$

$$\bar{r}\,' = (0, 0, z') = z'\hat{e}_z$$

$$d\bar{r}\,' = (0, 0, dz') = dz'\hat{e}_z \quad \Rightarrow \quad |d\bar{r}\,'| = dz'$$

$$\bar{r} = (x_P, y_P, z_P) \quad \Rightarrow \quad \bar{r} - \bar{r}\,' = (x_P, y_P, z_P - z') \quad \Rightarrow \quad |\bar{r} - \bar{r}\,'| = \sqrt{x_P^2 + y_P^2 + (z_P - z')^2}$$

$$\frac{|d\bar{r}\,'|}{|\bar{r} - \bar{r}\,'|} = \frac{dz'}{\sqrt{\rho_P^2 + (z_P - z')^2}}$$

where ρ_P is the distance of P from the curve C

Step 2. Calculate the integral $\phi(\bar{r}_P) = \frac{\lambda}{4\pi\varepsilon_0} \int_{-L}^{+L} \frac{dz'}{\sqrt{\rho_P^2 + (z_P - z')^2}}$

EXAMPLE: the force on a wire in a magnetic field

The force on a wire L carrying current I, in a magnetic field $\overline{B}$ is: $\overline{F} = I\int_L \left(d\overline{l} \times \overline{B}\right)$

Calculate the force on a circular coil L (with radius R and center in the origin). L lies on xy-plane (z=0) and has only one turn around the z-axis. The magnetic field $\overline{B}$ is defined in cylindrical coordinates by the expression: $\overline{B} = B_0\rho\left(\cos\varphi\,\hat{e}_z + \sin\varphi\,\hat{e}_\varphi\right)$

Use the following steps:

(a) express $d\overline{l}$ in a cylindrical coordinate system

(b) calculate $d\overline{l} \times \overline{B}$ in a cylindrical coordinate system

(c) Integrate and calculate $\overline{F}$ (here you can use a cartesian coordinate system)

$$d\overline{l} = \rho d\varphi\hat{e}_\varphi$$

$$d\overline{l} \times \overline{B} = \rho d\varphi\hat{e}_\varphi \times B_0\rho(\cos\varphi\hat{e}_z + \sin\varphi\hat{e}_\varphi) = B_0\rho^2\cos\varphi d\varphi\hat{e}_\rho$$

$$\overline{F} = I\int_0^{2\pi} B_0\rho^2\cos\varphi d\varphi\hat{e}_\rho = IB_0R^2\int_0^{2\pi}\cos\varphi\hat{e}_\rho d\varphi =$$

$$= IB_0R^2\int_0^{2\pi}\cos\varphi(\cos\varphi\hat{e}_x + \sin\varphi\hat{e}_y)d\varphi =$$

$$IB_0R^2\int_0^{2\pi}\cos^2\varphi\hat{e}_x d\varphi + IB_0R^2\int_0^{2\pi}\cos\varphi\sin\varphi\hat{e}_y d\varphi = IB_0R^2\hat{e}_x\int_0^{2\pi}\cos^2\varphi d\varphi =$$

$$= IB_0R^2\hat{e}_x\left[\frac{\varphi}{2} + \frac{\sin 2\varphi}{4}\right]_0^{2\pi} = IB_0R^2\pi\,\hat{e}_x$$