LEKTION, VECKA 6

Sketch of the solutions

DISCLAMER:

- **These are only a sketch of the solutions. For the details, you are supposed to attend "lektion" and ask directly to the teachers.**
- **I have written this file very quickly, copying from my notes.**
- **There might be mistakes (if you find an error, let me know).**
- **The language is poor: it is a mix of poor Swedish and english.**

PROBLEM 1

$$\iint_S \frac{r^3+1}{r^2}\hat{e}_r \cdot d\bar{S} = \iint_S \left(r+\frac{1}{r^2}\right)\hat{e}_r \cdot d\bar{S} = \underbrace{\iint_S r\hat{e}_r \cdot d\bar{S}}_{\substack{\text{vi kan använda} \\ \text{Gauss sats}}} + \underbrace{\iint_S \frac{1}{r^2}\hat{e}_r \cdot d\bar{S}}_{\substack{\text{vi kan använda} \\ \text{sats 17.5}}} =$$

$$= \iiint_V div\left(r\hat{e}_r\right)dV + 4\pi = \iiint_V 3dV + 4\pi = 3\frac{4}{3}\pi abc + 4\pi = 4\pi(1+abc)$$

PROBLEM 2

$$\iint_S \frac{r^2+1}{r}\hat{e}_\varphi \cdot d\bar{S} = \iint_{S+S_\varepsilon-S_\varepsilon} \frac{r^2+1}{r}\hat{e}_\varphi \cdot d\bar{S} = \iint_{S+S_\varepsilon} \frac{r^2+1}{r}\hat{e}_\varphi \cdot d\bar{S} + \iint_{-S_\varepsilon} \frac{r^2+1}{r}\hat{e}_\varphi \cdot d\bar{S} =$$

$$= \underbrace{\iint_{S+S_\varepsilon} \frac{r^2+1}{r}\hat{e}_\varphi \cdot d\bar{S}}_{\substack{\text{vi kan använder} \\ \text{Gauss sats}}} - \underbrace{\iint_{S_\varepsilon} \frac{r^2+1}{r}\hat{e}_\varphi \cdot d\bar{S}}_{\substack{S_\varepsilon \text{ är en sfär,} \\ \text{så vi kan beräkna} \\ \text{flödet direkt}}}$$

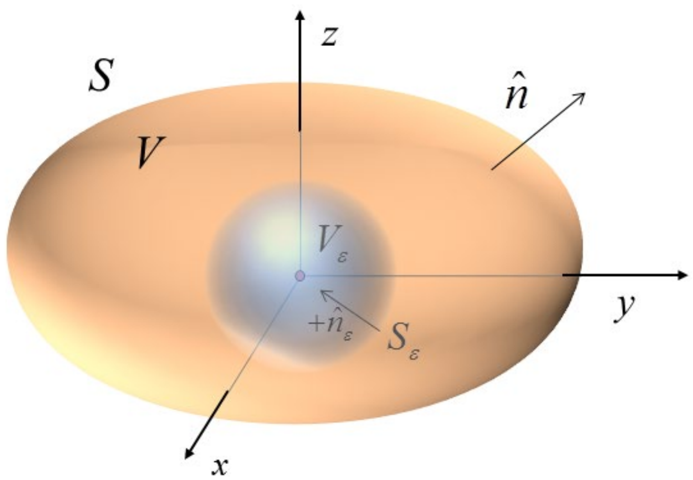
Nu, vi kan notera att (se uttryck (11.62))

$$\nabla \cdot \left(\frac{r^2+1}{r}\hat{e}_\varphi\right) = 0$$

och på S_ε vi har $d\bar{S} = r^2 sin\theta d\theta d\varphi \hat{e}_r$ (med $r = \varepsilon$), se uttryck (11.59). Så,

$$\iint_S \frac{r^2+1}{r}\hat{e}_\varphi \cdot d\bar{S} = \iiint_{V-V_\varepsilon} \underbrace{\nabla \cdot \left(\frac{r^2+1}{r}\hat{e}_\varphi\right)}_{=0} dV - \iint_{S_\varepsilon} \frac{r^2+1}{r}\hat{e}_\varphi \cdot \left(r^2 \sin\theta d\theta d\varphi \hat{e}_r\right) =$$

$$= -\iint_{S_\varepsilon} \frac{r^2+1}{r} r^2 \sin\theta \underbrace{\hat{e}_\varphi \cdot \hat{e}_r}_{=0} d\theta d\varphi = 0$$



PROBLEM 3

$$\int_L \bar{A}\cdot d\bar{r} = \int_L \frac{\omega\rho+\omega}{\rho}\hat{e}_\varphi\cdot d\bar{r} = \int_L\left(\omega+\frac{\omega}{\rho}\right)\hat{e}_\varphi\cdot d\bar{r} = \underbrace{\int_L \omega\hat{e}_\varphi\cdot d\bar{r}}_{\substack{\text{vi kan använder}\\ \text{Stokes stats}}} + \underbrace{\int_L \frac{\omega}{\rho}\hat{e}_\varphi\cdot d\bar{r}}_{\substack{\text{vi kan använder}\\ \text{stats (17.28)}}} =$$

$$= \iint_S \nabla\times\left(\hat{e}_\varphi\right)\cdot d\bar{S} - 2\pi\omega = \iint_S \frac{\omega}{\rho}\hat{e}_z\cdot d\bar{S} - 2\pi\omega$$

Notera att orientering av L i punkten $x=a,\ y=0,\ z=0$ är $-\hat{e}_y$. Så kurvan genomlöps i negativa $\hat{e}_\varphi$-riktningen och därför, $N=-1$ i (17.28).
Kurvan är en cirkel, med centrum i origo, så $d\bar{S} = -\rho d\rho d\varphi \hat{e}_z$. Så, integralen blir

$$\int_L \bar{A}\cdot d\bar{r} = \iint_S \frac{\omega}{\rho}\underbrace{\hat{e}_z\cdot(-\hat{e}_z)}_{=-1}\rho d\rho d\varphi - 2\pi\omega = -2\pi\omega\int_0^a d\rho - 2\pi\omega = -2\pi\omega a - 2\pi\omega = -2\pi\omega(a+1)$$

PROBLEM 4

$$\bar{A} = \frac{\rho^2+1}{\rho}\hat{e}_\varphi + \rho^2\hat{e}_\rho = \rho\hat{e}_\varphi + \rho^2\hat{e}_\rho + \frac{1}{\rho}\hat{e}_\varphi$$

Så, integralen blir

$$\oint_L \bar{A}\cdot d\bar{r} = \underbrace{\oint_L\left(\rho\hat{e}_\varphi+\rho^2\hat{e}_\rho\right)\cdot d\bar{r}}_{\text{vi kan använder Stokes sats}} + \underbrace{\oint_L \frac{1}{\rho}\hat{e}_\varphi\cdot d\bar{r}}_{-2\pi} = \iint_S \nabla\times\left(\rho\hat{e}_\varphi+\rho^2\hat{e}_\rho\right)\cdot d\bar{S} - 2\pi$$

Notera att I punkten $x=0,\ y=1,\ z=0$ har L tangentvektor $\hat{e}_x$. Så kurvan genomlöps i negativa $\hat{e}_\varphi$-riktningen och därför $N=-1$ i (17.28).
Nu, vi kan beräkna rotatione med uttryck (11.45)

$$\nabla\times\left(\rho\hat{e}_\varphi+\rho^2\hat{e}_\rho\right) = \left(\frac{\partial\rho}{\partial z}\right)\hat{e}_\rho + \left(\frac{\partial\rho^2}{\partial z}\right)\hat{e}_\varphi + \left(\frac{\partial\left(\rho^2\right)}{\rho\partial\rho} - \frac{\partial\rho^2}{\rho\partial\varphi}\right)\hat{e}_z = 2\hat{e}_z$$

Så, därför $d\bar{S} = dS\hat{e}_z$, vi har blir

$$\iint_S \nabla \times \left(\rho\hat{e}_\varphi + \rho^2\hat{e}_\rho\right)\cdot d\bar{S} = \iint_S 2\hat{e}_z \cdot (-dS\hat{e}_z) = -2\iint_S dS = -2\pi ab$$

där vi har använt att area av ellipses är πab.
Så, integralen blir

$$\oint_L \bar{A}\cdot d\bar{r} = -2\pi ab - 2\pi = -2\pi(1+ab)$$

PROBLEM 5

Vi måste lösa Poisson ekvationen, se (18.3),

$$\nabla^2 V = -\frac{\rho_c}{\varepsilon_0}$$

Dessotum, vi kan beräkna det elektriska fältet från potentialen med hjälp av

$$\bar{E} = -\nabla V\,.$$

Due to to the symmetry of the problem and to fact that the charge density is constant (i.e. does not depend on z and φ), the solution will depend only on ρ. Therefore, the derivatives in z and φ of the Laplacian are zero.
So, if we express the Laplacian in a cylindrical coordinate system, expression (15.4), we obtain

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\underbrace{\frac{\partial^2 V}{\partial\varphi^2}}_{=0} + \underbrace{\frac{\partial^2 V}{\partial z^2}}_{=0} = -\frac{\rho_c}{\varepsilon_0}$$

And for the electric field, using (11.43) for the gradient in cylindrical coordinate, we have

$$\bar{E} = -\nabla V = -\frac{\partial V}{\partial\rho}\hat{e}_\rho - \frac{1}{\rho}\underbrace{\frac{\partial V}{\partial\varphi}}_{=0}\hat{e}_\varphi - \underbrace{\frac{\partial V}{\partial z}}_{=0}\hat{e}_z = -\frac{\partial V}{\partial\rho}\hat{e}_\rho$$

So, we have to solve the equation

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) = -\frac{\rho_c}{\varepsilon_0}$$

We start to solve the equation inside the cylinder.

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dV(\rho)}{d\rho}\right) = -\frac{\rho_0}{\varepsilon_0}\left(1-\frac{\rho}{R_0}\right) \quad\Rightarrow\quad \frac{d}{d\rho}\left(\rho\frac{dV(\rho)}{d\rho}\right) = -\frac{\rho_0}{\varepsilon_0}\left(\rho-\frac{\rho^2}{R}\right) \quad\Rightarrow$$

$$\Rightarrow\quad \frac{dV(\rho)}{d\rho} = -\frac{\rho_0}{\varepsilon_0}\left(\frac{\rho}{2}-\frac{\rho^2}{3R}\right) + \frac{c}{\rho}$$

But now can note that we already have the expression for the electric field,

$$\bar{E} = -\frac{\partial V}{\partial\rho}\hat{e}_\rho = \left(\frac{\rho_0}{\varepsilon_0}\left(\frac{\rho}{2}-\frac{\rho^2}{3R}\right) - \frac{c}{\rho}\right)\hat{e}_\rho$$

On the cylinder axis, the field must be zero. Se we have *c=0*.
Now we can continue to solve the Poisson equation. Integrating in ρ we obtain,

$$\frac{\partial V}{\partial\rho} = -\frac{\rho_0}{\varepsilon_0}\left(\frac{\rho}{2}-\frac{\rho^2}{3R}\right) \quad\Rightarrow\quad V(\rho) = -\frac{\rho_0}{\varepsilon_0}\left(\frac{\rho^2}{4}-\frac{\rho^3}{9R}\right) + a$$

Therefore, inside the cylinder the potential and the field are

$$V_{in} = -\frac{\rho_0}{\varepsilon_0}\left(\frac{\rho^2}{4} - \frac{\rho^3}{9R}\right) + a$$

$$\bar{E}_{in} = \frac{\rho_0}{\varepsilon_0}\left(\frac{\rho}{2} - \frac{\rho^2}{3R}\right)\hat{e}_\rho$$

Outside the cylinder, the charge density is zero, so we are left with the Laplace equation in cylindrical symmetry. We already know from Section 18.2, expression (18.10) that in this case the solution is

$$V_{out}(\rho) = c\ln\rho + d$$

The electric field is

$$\bar{E}_{out} = -\frac{\partial V}{\partial \rho}\hat{e}_\rho = -\frac{c}{\rho}\hat{e}_\rho$$

Now we need to find the integration constant *a, c and d*. We have three condition to use.

(1) Continuity of the electric field at $\rho = R$

$$\bar{E}_{in}(R) = \bar{E}_{out}(R)$$

(2) The value of the electrostatic potential at $\rho = R$

$$V_{out}(R) = V_0$$

(3) Continuity of the potential at $\rho = R$

$$V_{in}(R) = V_{out}(R)$$

These three conditions lead to a system of three equations in the three unknown *a, c and d*,

$$\begin{cases} \frac{\rho_0}{\varepsilon_0}\left(\frac{R}{2} - \frac{R^2}{3R}\right) = -\frac{c}{R} \\ c\ln R + d = V_0 \\ -\frac{\rho_0}{\varepsilon_0}\left(\frac{R^2}{4} - \frac{R^3}{9R}\right) + a = c\ln R + d \end{cases}$$

If we solve this system, we obtain

$$a = V_0 + \frac{5\rho_0}{36\varepsilon_0}R^2 \qquad c = -\frac{\rho_0}{6\varepsilon_0}R^2 \qquad d = V_0 + \frac{\rho_0}{6\varepsilon_0}R^2\ln R$$

Finally, inserting these three expressions in to the expressions of the electric field and potential, we obtain

$$V_{in}(\rho) = -\frac{\rho_0}{\varepsilon_0}\left(\frac{\rho^2}{4} - \frac{\rho^3}{9R}\right) + V_0 + \frac{5\rho_0}{36\varepsilon_0}R^2$$

$$V_{out}(\rho) = V_0 - \frac{\rho_0}{6\varepsilon_0}R^2\ln\frac{\rho}{R}$$

$$\bar{E}_{in}(\rho) = \frac{\rho_0}{\varepsilon_0}\left(\frac{\rho}{2} - \frac{\rho^2}{3R}\right)\hat{e}_\rho$$

$$\bar{E}_{out}(\rho) = \frac{\rho_0}{6\varepsilon_0}\frac{R^2}{\rho}\hat{e}_\rho$$

PROBLEM 6

$$\nabla^2 V=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\underbrace{\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)}_{=0}+\frac{1}{r^2\sin^2\theta}\underbrace{\frac{\partial^2 V}{\partial\varphi^2}}_{=0}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)=-\frac{\rho_c}{\varepsilon_0}$$

And for the electric field, using (11.61) for the gradient in spherical coordinate, we have

$$\bar{E}=-\nabla V=-\frac{\partial V}{\partial r}\hat{e}_r-\frac{1}{r}\frac{\partial V}{\partial\theta}\hat{e}_\theta-\frac{1}{r\sin\theta}\frac{\partial V}{\partial\varphi}\hat{e}_\varphi=-\frac{\partial V}{\partial r}\hat{e}_r$$

So, we have to solve the equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)=-\frac{\rho_0}{\varepsilon_0}\left(\frac{r}{R}\right)^\alpha$$

We start to solve the equation inside the cylinder.

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV(r)}{dr}\right)=-\frac{\rho_0}{\varepsilon_0}\left(\frac{r}{R}\right)^\alpha\quad\Rightarrow\quad\frac{d}{dr}\left(r^2\frac{dV(r)}{dr}\right)=-\frac{\rho_0}{\varepsilon_0}\frac{r^{\alpha+2}}{R^\alpha}$$

$$\Rightarrow\quad r^2\frac{dV(r)}{dr}=-\frac{\rho_0}{\varepsilon_0}\frac{1}{\alpha+3}\frac{r^{\alpha+3}}{R^\alpha}+c\quad\Rightarrow\quad\frac{dV(r)}{dr}=-\frac{\rho_0}{\varepsilon_0}\frac{1}{\alpha+3}\frac{r^{\alpha+1}}{R^\alpha}+\frac{c}{r^2}$$

But now can note that we already have the expression for the electric field,

$$\bar{E}=-\frac{\partial V}{\partial r}\hat{e}_r=\left(\frac{\rho_0}{\varepsilon_0}\frac{1}{\alpha+3}\frac{r^{\alpha+1}}{R^\alpha}-\frac{c}{r^2}\right)\hat{e}_r$$

In the center of the sphere, the field must be zero. Se we have $c=0$.
Now we can continue to solve the Poisson equation. Integrating in r we obtain,

$$\frac{dV(r)}{dr}=-\frac{\rho_0}{\varepsilon_0}\frac{1}{\alpha+3}\frac{r^{\alpha+1}}{R^\alpha}\quad\Rightarrow\quad V(r)=-\frac{\rho_0}{\varepsilon_0}\frac{1}{(\alpha+3)(\alpha+2)}\frac{r^{\alpha+2}}{R^\alpha}+d$$

Therefore, inside the sphere the potential and the field are

$$V_{in}(r)=-\frac{\rho_0}{\varepsilon_0}\frac{1}{(\alpha+3)(\alpha+2)}\frac{r^{\alpha+2}}{R^\alpha}+d$$

$$\bar{E}_{in}(r)=\frac{\rho_0}{\varepsilon_0}\frac{1}{\alpha+3}\frac{r^{\alpha+1}}{R^\alpha}\hat{e}_r$$

Outside the sphere, the charge density is zero, so we are left with the Laplace equation in spherical symmetry. We already know from Section 18.2, expression (18.12) that in this case the solution is

$$V_{out}(r)=-\frac{a}{r}+b$$

Note that we have the problem specify that we can assume $\lim_{r\to\infty}V(r)=0$. This implies $b=0$.
Now we can calculate the electric field with

$$\bar{E}_{out}=-\frac{\partial V}{\partial r}\hat{e}_r=-\frac{a}{r^2}\hat{e}_r$$

We are left with the determination of the constant a and d.

We have two conditions to use.

(1) Continuity of the electric field at $r=R$

$$\bar{E}_{in}(R) = \bar{E}_{out}(R)$$

(2) Continuity of the potential at $r = R$

$$V_{in}(R) = V_{out}(R)$$

These two conditions lead to a system of two equations for determining the two remaining constants.

$$\begin{cases} \dfrac{\rho_0}{\varepsilon_0}\dfrac{1}{\alpha+3}\dfrac{R^{\alpha+1}}{R^\alpha} = -\dfrac{a}{R^2} \\ -\dfrac{\rho_0}{\varepsilon_0}\dfrac{1}{(\alpha+3)(\alpha+2)}\dfrac{R^{\alpha+2}}{R^\alpha} + d = -\dfrac{a}{R} \end{cases}$$

If we solve this system, we obtain

$$a = -\frac{\rho_0}{\varepsilon_0}\frac{R^3}{\alpha+3} \qquad d = \frac{\rho_0}{\varepsilon_0}\frac{R^2}{\alpha+2}$$

Finally, inserting these three expressions in to the expressions of the electric field and potential, we obtain

$$V_{in}(r) = \frac{\rho_0}{\varepsilon_0}\frac{R^2}{\alpha+2}\left(1 - \frac{1}{(\alpha+3)}\frac{r^{\alpha+2}}{R^{\alpha+2}}\right)$$

$$V_{out}(r) = \frac{\rho_0}{\varepsilon_0}\frac{R^3}{\alpha+3}\frac{1}{r}$$

$$\bar{E}_{in}(r) = \frac{\rho_0}{\varepsilon_0}\frac{1}{\alpha+3}\frac{r^{\alpha+1}}{R^\alpha}\hat{e}_r$$

$$\bar{E}_{out}(r) = \frac{\rho_0}{\varepsilon_0}\frac{1}{\alpha+3}\frac{R^3}{r^2}\hat{e}_r$$

Note that,

$$Q = \int \rho(r)dV = \rho_0\frac{4\pi R^3}{\alpha+3}$$

So, we can also write

$$V_{out}(r) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}$$

$$\bar{E}_{out}(r) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{e}_r$$

which are the same expressions of the potential and field produced by a point charge.