Lektion, VECKA 4

Sketch of the solutions

DISCLAMER:

- **These are only a sketch of the solutions. For the details, you are supposed to attend "lektion" and ask directly to the teachers.**
- **I have written this file very quickly, copying from my notes.**
- **There might be mistakes (if you find an error, let me know).**
- **The language is poor: it is a mix of poor Swedish and english.**

PROBLEM (1)

From Section 10.8 of the book, we have:

$$grad\phi = \frac{\partial \phi}{\partial \rho}\hat{e}_\rho + \frac{1}{\rho}\frac{\partial \phi}{\partial \varphi}\hat{e}_\varphi + \frac{\partial \phi}{\partial z}\hat{e}_z$$

$$div\overline{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$

$$rot\overline{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{e}_\rho + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{e}_\varphi + \left(\frac{1}{\rho}\frac{\partial \rho A_\varphi}{\partial \rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \varphi}\right)\hat{e}_z$$

So, applying these expressions, we get:

(a) $$grad\left(\rho^2\sin\varphi + \frac{z\sin\varphi}{\rho}\right) =$$

$$= \frac{\partial}{\partial \rho}\left(\rho^2\sin\varphi + \frac{z\sin\varphi}{\rho}\right)\hat{e}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \varphi}\left(\rho^2\sin\varphi + \frac{z\sin\varphi}{\rho}\right)\hat{e}_\varphi + \frac{\partial}{\partial z}\left(\rho^2\sin\varphi + \frac{z\sin\varphi}{\rho}\right)\hat{e}_z =$$

$$= \left(2\rho\sin\varphi - \frac{z\sin\varphi}{\rho^2}\right)\hat{e}_\rho + \left(\rho\cos\varphi + \frac{z\cos\varphi}{\rho^2}\right)\hat{e}_\varphi + \frac{\sin\varphi}{\rho}\hat{e}_z$$

(b) $\overline{A} = \hat{e}_\rho \quad \Rightarrow \quad A_\rho = 1,\ A_\varphi = 0,\ A_z = 0$

$$div\left(\hat{e}_\rho\right) = \frac{1}{\rho}\frac{\partial(\rho)}{\partial \rho} = \frac{1}{\rho}$$

(c) $\overline{A} = \hat{e}_\varphi \quad \Rightarrow \quad A_\rho = 0,\ A_\varphi = 1,\ A_z = 0$

$$div\left(\hat{e}_\varphi\right)=\frac{1}{\rho}\frac{\partial(\rho 0)}{\partial\rho}+\frac{1}{\rho}\frac{\partial 1}{\partial\varphi}+\frac{\partial 0}{\partial z}=0$$

(d) $\overline{A}=\rho\cos\varphi\hat{e}_\rho+\rho\sin\varphi\hat{e}_\varphi+z^2\hat{e}_z$

$$\Rightarrow\quad A_\rho=\rho\cos\varphi,\ A_\varphi=\rho\sin\varphi,\ A_z=z^2$$

$$div\left(\rho\cos\varphi\hat{e}_\rho+\rho\sin\varphi\hat{e}_\varphi+z^2\hat{e}_z\right)=\frac{1}{\rho}\frac{\partial\left(\rho^2\cos\varphi\right)}{\partial\rho}+\frac{1}{\rho}\frac{\partial\rho\sin\varphi}{\partial\varphi}+\frac{\partial z^2}{\partial z}=$$

$$=2\cos\varphi+\cos\varphi+2z=3\cos\varphi+2z$$

(e) $\overline{A}=\hat{e}_\varphi\quad\Rightarrow\quad A_\rho=0,\ A_\varphi=1,\ A_z=0$

$$rot\left(\hat{e}_\varphi\right)=\left(\frac{1}{\rho}\frac{\partial 0}{\partial\varphi}-\frac{\partial 1}{\partial z}\right)\hat{e}_\rho+\left(\frac{\partial 0}{\partial z}-\frac{\partial 0}{\partial\rho}\right)\hat{e}_\varphi+\left(\frac{1}{\rho}\frac{\partial\rho}{\partial\rho}-\frac{1}{\rho}\frac{\partial 0}{\partial\varphi}\right)\hat{e}_z=\frac{1}{\rho}\hat{e}_z$$

PROBLEM (2)

From Section 10.8 of the book, we have:

$$grad\phi=\frac{\partial\phi}{\partial r}\hat{e}_r+\frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{e}_\theta+\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\hat{e}_\varphi$$

$$div\overline{A}=\frac{1}{r^2}\frac{\partial\left(r^2A_r\right)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta A_\theta\right)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$$

$$rot\overline{A}=\frac{1}{r\sin\theta}\left(\frac{\partial\left(\sin\theta A_\varphi\right)}{\partial\theta}-\frac{\partial A_\theta}{\partial\varphi}\right)\hat{e}_r+\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi}-\frac{\partial\left(rA_\varphi\right)}{\partial r}\right)\hat{e}_\theta+\frac{1}{r}\left(\frac{\partial\left(rA_\theta\right)}{\partial r}-\frac{\partial A_r}{\partial\theta}\right)\hat{e}_\varphi$$

So, applying these expressions, we get:

(a) $grad\left(r^2\right)=\frac{\partial r^2}{\partial r}\hat{e}_r+\frac{1}{r}\frac{\partial r^2}{\partial\theta}\hat{e}_\theta+\frac{1}{r\sin\theta}\frac{\partial r^2}{\partial\varphi}\hat{e}_\varphi=2r\hat{e}_r=2\overline{r}$

(b) $grad\left(\frac{\cos\theta}{r^2}\right)=\frac{\partial}{\partial r}\left(\frac{\cos\theta}{r^2}\right)\hat{e}_r+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{r^2}\right)\hat{e}_\theta+\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(\frac{\cos\theta}{r^2}\right)\hat{e}_\varphi=$

$$=-\frac{2\cos\theta}{r^3}\hat{e}_r-\frac{\sin\theta}{r^3}\hat{e}_\theta$$

(c) $\overline{A}=\overline{r}=r\hat{e}_r\quad\Rightarrow\quad A_r=r,\ A_\theta=0,\ A_\varphi=0$

$$div\left(\bar{r}\right)=\frac{1}{r^2}\frac{\partial\left(r^3\right)}{\partial r}=3$$

(d) $\bar{A}=r^2\sin\theta\hat{e}_r+\cos\varphi\hat{e}_\varphi \quad\Rightarrow\quad A_r=r^2\sin\theta,\, A_\theta=0,\, A_\varphi=\cos\varphi$

$$rot\left(r^2\sin\theta\hat{e}_r+\cos\varphi\hat{e}_\varphi\right)=$$

$$=\frac{1}{r\sin\theta}\left(\frac{\partial\left(\sin\theta\cos\varphi\right)}{\partial\theta}\right)\hat{e}_r+\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial r^2\sin\theta}{\partial\varphi}-\frac{\partial\left(r\cos\varphi\right)}{\partial r}\right)\hat{e}_\theta+\frac{1}{r}\left(-\frac{\partial r^2\sin\theta}{\partial\theta}\right)\hat{e}_\varphi=$$

$$=\frac{\cos\theta\cos\varphi}{r\sin\theta}\hat{e}_r-\frac{\cos\varphi}{r}\hat{e}_\theta-r\cos\theta\hat{e}_\varphi$$

PROBLEM (3)

$$\hat{n}=-\hat{e}_r$$

$$dS=r^2\sin\theta d\theta d\varphi \quad \text{men} \quad r=1 \quad \text{så} \quad dS=\sin\theta d\theta d\varphi$$

$$\iint_S \hat{e}_\varphi\times\hat{n}dS=-\iint_S \underbrace{\hat{e}_\varphi\times\hat{e}_r}_{\hat{e}_\theta}\sin\theta d\theta d\varphi=-\iint_S \hat{e}_\theta\sin\theta d\theta d\varphi=$$

$$=-\int_0^{\pi/2}\int_0^{\pi/2}\left(\cos\theta\cos\varphi\hat{e}_x+\cos\theta\sin\varphi\hat{e}_y-\sin\theta\hat{e}_z\right)\sin\theta d\theta d\varphi$$

$$=-\frac{1}{2}\hat{e}_x-\frac{1}{2}\hat{e}_y+\frac{\pi^2}{8}\hat{e}_z$$

PROBLEM (4)

$$\int_P^Q\left(\frac{1}{r}\hat{e}_r+\frac{1}{r\sin\theta}\hat{e}_\varphi\right)\cdot d\bar{r}$$

(a)

A line integral $\int_L \bar{A}\cdot d\bar{r}$ is independent of the integration path if the field $\bar{A}$ is conservative (and hence, if it has a potential).
From Section 9.4 of the book, the field has a potential if $\nabla\times\bar{A}=0$.

$$\nabla\times\left(\frac{1}{r}\hat{e}_r+\frac{1}{r\sin\theta}\hat{e}_\varphi\right)=$$

$$\frac{1}{r\sin\theta}\left(\frac{\partial\left(\sin\theta\frac{1}{r\sin\theta}\right)}{\partial\theta}\right)\hat{e}_r+\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\left(\frac{1}{r}\right)-\frac{\partial\left(r\frac{1}{r\sin\theta}\right)}{\partial r}\right)\hat{e}_\theta+\frac{1}{r}\left(-\frac{\partial}{\partial\theta}\left(\frac{1}{r}\right)\right)\hat{e}_\varphi=0$$

(b) If vector field has a potential, $\bar{A} = grad\phi$, then

$$\int_P^Q \bar{A} \cdot d\bar{r} = \phi(q) - \phi(p)$$

The potential is:

$\phi = \ln(r) + \varphi + c$,

In P: r=1 and $\varphi = \frac{\pi}{2}$ and in Q: r=3, $\varphi = \frac{3\pi}{2}$
So,

$$\int_P^Q \bar{A} \cdot d\bar{r} = \ln(3) + \frac{3\pi}{2} - \ln(1) - \frac{\pi}{2} = ln3 + \pi$$

Note that the fact that the path does not go through the phi=pi plane implies that the path has only one turn around the z-axis. This is because if the path can do more than one turn, you will get an integral that is dependent on the number of turns, which means that does not simply depend on the the initial point P and the final point Q.

PROBLEM (5)

$$\bar{A} = \left(\frac{\cos 2\theta}{r^3} \hat{e}_\theta - \frac{\sin 2\theta}{r^3} \hat{e}_r \right)$$

(a) $$rot\bar{A} = \frac{1}{r\sin\theta}\left(-\frac{\partial\left(\frac{\cos 2\theta}{r^3}\right)}{\partial\varphi} \right)\hat{e}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial\left(-\frac{\sin 2\theta}{r^3}\right)}{\partial\varphi} \right)\hat{e}_\theta + \frac{1}{r}\left(\frac{\partial\left(r\frac{\cos 2\theta}{r^3}\right)}{\partial r} - \frac{\partial\left(-\frac{\sin 2\theta}{r^3}\right)}{\partial\theta} \right)\hat{e}_\varphi = \bar{0}$$

(b) $$div\bar{A} = \frac{1}{r^2}\frac{\partial\left(r^2\left(-\frac{\sin 2\theta}{r^3}\right)\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta\frac{\cos 2\theta}{r^3}\right)}{\partial\theta} = \frac{\cos 3\theta}{r^4\sin\theta}$$

(c) $\nabla \times \bar{A} = 0$ so it has a potential $\bar{A} = grad\phi$ with $\phi = \frac{\sin 2\theta}{2r^2} + c$

PROBLEM (6)

$$\nabla^2\hat{e}_\varphi = \nabla\left(\nabla\cdot\hat{e}_\varphi\right) - \nabla\times\left(\nabla\times\hat{e}_\varphi\right)$$

$$\nabla\cdot\hat{e}_\varphi = 0$$

$$\nabla \times \hat{e}_\varphi = \frac{1}{\rho}\hat{e}_z \quad \Rightarrow \quad \nabla \times \left(\nabla \times \hat{e}_\varphi\right) = \nabla \times \left(\frac{\hat{e}_z}{\rho}\right) = \frac{\hat{e}_\varphi}{\rho^2}$$

So:

$$\nabla^2 \hat{e}_\varphi = -\frac{1}{\rho^2}\hat{e}_\varphi$$