LEKTION, VECKA 5

Sketch of the solutions

DISCLAMER:

- **These are only a sketch of the solutions. For the details, you are supposed to attend “lektion” and ask directly to the teachers.**
- **I have written this file very quickly, copying from my notes.**
- **There might be mistakes (if you find an error, let me know).**
- **The language is poor: it is a mix of poor Swedish and english.**

PROBLEM 1

(a)

$$\left(\overline{B}\cdot\nabla\right)\left(\phi\overline{A}\right)=\left(\overline{B}\cdot\nabla\right)\left(\underset{\cdot}{\phi}\overline{A}\right)+\left(\overline{B}\cdot\nabla\right)\left(\phi\underset{\cdot}{\overline{A}}\right)=$$

$$=\left\{\left(\overline{b}\cdot\overline{n}\right)\left(\underset{\cdot}{c}\overline{a}\right)+\left(\overline{b}\cdot\overline{n}\right)\left(c\underset{\cdot}{\overline{a}}\right)=\left(\overline{b}\cdot\overline{n}\underset{\cdot}{c}\right)\left(\overline{a}\right)+c\left(\overline{b}\cdot\overline{n}\right)\left(\underset{\cdot}{\overline{a}}\right)=\overline{a}\left(\overline{b}\cdot\overline{n}\underset{\cdot}{c}\right)+c\left(\overline{b}\cdot\overline{n}\right)\underset{\cdot}{\overline{a}}\right\}=$$

$$=\overline{A}\left(\overline{B}\cdot\nabla\underset{\cdot}{\phi}\right)+\phi\left(\overline{B}\cdot\nabla\right)\underset{\cdot}{\overline{A}}=\overline{A}\left(\overline{B}\cdot\nabla\phi\right)+\phi\left(\overline{B}\cdot\nabla\right)\overline{A}$$

(b) $$\left(\overline{B}\cdot\nabla\right)\left(\overline{A}\times\overline{B}\right)=-\overline{B}\times\left(\overline{B}\cdot\nabla\right)\overline{A}+\overline{A}\times\left(\overline{B}\cdot\nabla\right)\overline{B}$$

$$\left(\overline{B}\cdot\nabla\right)\left(\overline{A}\times\overline{B}\right)=\left(\overline{B}\cdot\nabla\right)\left(\underset{\cdot}{\overline{A}}\times\overline{B}\right)+\left(\overline{B}\cdot\nabla\right)\left(\overline{A}\times\underset{\cdot}{\overline{B}}\right)=$$

$$=\left\{\underbrace{\left(\overline{b}\cdot\overline{n}\right)}_{=c}\left(\underset{\cdot}{\overline{a}}\times\overline{b}\right)+\underbrace{\left(\overline{b}\cdot\overline{n}\right)}_{=c}\left(\overline{a}\times\underset{\cdot}{\overline{b}}\right)=\left(\left(c\underset{\cdot}{\overline{a}}\right)\times\overline{b}\right)+\left(\overline{a}\times\left(c\underset{\cdot}{\overline{b}}\right)\right)=-\overline{b}\times\left(c\underset{\cdot}{\overline{a}}\right)+\overline{a}\times\left(c\underset{\cdot}{\overline{b}}\right)=\right.$$

$$\left.=-\overline{b}\times\left(\left(\overline{b}\cdot\overline{n}\right)\underset{\cdot}{\overline{a}}\right)+\overline{a}\times\left(\left(\overline{b}\cdot\overline{n}\right)\underset{\cdot}{\overline{b}}\right)\right\}=$$

$$=-\overline{B}\times\left(\overline{B}\cdot\nabla\right)\overline{A}+\overline{A}\times\left(\overline{B}\cdot\nabla\right)\overline{B}$$

PROBLEM 2

(a) $\nabla r=\hat{e}_r$

$$\nabla r=\frac{\partial r}{\partial r}\hat{e}_r+\frac{1}{r}\frac{\partial r}{\partial\theta}\hat{e}_\theta+\frac{1}{r\sin\theta}\frac{\partial r}{\partial\varphi}\hat{e}_\varphi=\frac{\partial r}{\partial r}\hat{e}_r=\hat{e}_r$$

$(b) \quad \nabla \cdot \bar{r} = 3$

$$\nabla \cdot \bar{r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta 0\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(0) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\right) = \frac{3r^2}{r^2} = 3$$

$(c) \quad \nabla \times \bar{r} = \bar{0}$

$$\nabla \times \bar{r} = \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta 0\right) - \frac{1}{r\sin\theta}\frac{\partial 0}{\partial\varphi}\right)\hat{e}_r + \left(\frac{1}{r\sin\theta}\frac{\partial r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}(r0)\right)\hat{e}_\theta + \left(\frac{1}{r}\frac{\partial}{\partial r}(r0) - \frac{1}{r}\frac{\partial r}{\partial\theta}\right)\hat{e}_\varphi = \bar{0}$$

PROBLEM 3

Visa med indexräkning att:

(a) $\nabla \cdot \left(\bar{r} \times \nabla\phi\right) = 0$

$$\left.\begin{aligned} &\bar{v} = \bar{r} \times \nabla\phi \\ &\nabla \cdot \bar{v} = v_{i,i} \\ &v_i = \left(\bar{r} \times \nabla\phi\right)_i = \varepsilon_{ijk} r_j \left(\nabla\phi\right)_k \end{aligned}\right\} \Rightarrow \nabla \cdot \left(\bar{r} \times \nabla\phi\right) = \left(\varepsilon_{ijk} r_j \left(\nabla\phi\right)_k\right)_{,i} = \varepsilon_{ijk} r_{j,i} \left(\nabla\phi\right)_k + \varepsilon_{ijk} r_j \left(\nabla\phi\right)_{k,i}$$

Men $r_{i,j} \neq 0$ bara om $i = j$. Men om $i = j$, så $\varepsilon_{ijk} = 0$. Så, den fårsta termen är noll.
För den andra termen, vi kan notera att $\varepsilon_{ijk} = -\varepsilon_{jik}$. Så

$$\varepsilon_{ijk} r_j \left(\nabla\phi\right)_{k,i} = -r_j \varepsilon_{jik} \left(\nabla\phi\right)_{k,i} = -r_j \left[\nabla \times \nabla\phi\right]_j = -\bar{r} \cdot \left(\nabla \times \nabla\phi\right)$$

Men från Kapitel 12 vi känner att $\nabla \times \nabla\phi$=0. Så

$$\nabla \cdot \left(\bar{r} \times \nabla\phi\right) = 0$$

(b) $\nabla \cdot \left(\nabla\psi \times \nabla\phi\right) = 0$

$$\left.\begin{aligned} &\bar{v} = \nabla\psi \times \nabla\phi \\ &\nabla \cdot \bar{v} = v_{i,i} \\ &v_i = \left(\nabla\psi \times \nabla\phi\right)_i = \varepsilon_{ijk} \left(\nabla\psi\right)_j \left(\nabla\phi\right)_k \end{aligned}\right\} \Rightarrow \nabla \cdot \left(\nabla\psi \times \nabla\phi\right) = \left(\varepsilon_{ijk} \left(\nabla\psi\right)_j \left(\nabla\phi\right)_k\right)_{,i} =$$

$$= \varepsilon_{ijk} \left(\nabla\psi\right)_{j,i} \left(\nabla\phi\right)_k + \varepsilon_{ijk} \left(\nabla\psi\right)_j \left(\nabla\phi\right)_{k,i} =$$

(vi kan notera att $\varepsilon_{ijk} = \varepsilon_{kji}$ $\varepsilon_{ijk} = -\varepsilon_{jik}$. Så)

$$\begin{aligned} &= \varepsilon_{kij} \left(\nabla\psi\right)_{j,i} \left(\nabla\phi\right)_k - \varepsilon_{jik} \left(\nabla\psi\right)_j \left(\nabla\phi\right)_{k,i} = \\ &= \left(\nabla \times \left(\nabla\psi\right)\right)_k \left(\nabla\phi\right)_k - \left(\nabla\psi\right)_j \left(\nabla \times \left(\nabla\phi\right)\right)_j = \\ &= \left(\nabla \times \left(\nabla\psi\right)\right) \cdot \nabla\phi - \nabla\psi \cdot \left(\nabla \times \left(\nabla\phi\right)\right) = 0 \end{aligned}$$

därför att rotationen av gradienten är noll.

PROBLEM 4

(a) $\nabla(\bar{a}\cdot\bar{r})=\bar{a}$

Vi kan använda (11.26):

$$\nabla(\bar{a}\cdot\bar{r})=(\bar{r}\cdot\nabla)\bar{a}+(\bar{a}\cdot\nabla)\bar{r}+\bar{r}\times(\nabla\times\bar{a})+\bar{a}\times(\nabla\times\bar{r})=$$

={Alla termer med derivator av $\bar{a}$ är noll eftersom $\bar{a}$ är en konstant vektor. Kvar blir}=

$$=(\bar{a}\cdot\nabla)\bar{r}+\bar{a}\times(\nabla\times\bar{r})=$$

={Men $\nabla\times\bar{r}=\bar{0}$ som framgår av (10.63), vilket ger} =

$$=(\bar{a}\cdot\nabla)\bar{r}=\left(A_x\frac{\partial}{\partial x}+A_y\frac{\partial}{\partial y}+A_z\frac{\partial}{\partial z}\right)\bar{r}=$$

$$=a_x\underbrace{\frac{\partial\bar{r}}{\partial x}}_{=\hat{e}_x}+a_y\underbrace{\frac{\partial\bar{r}}{\partial y}}_{=\hat{e}_y}+a_z\underbrace{\frac{\partial\bar{r}}{\partial z}}_{=\hat{e}_z}=a_x\hat{e}_x+a_y\hat{e}_y+a_z\hat{e}_z=\bar{a}$$

(b) $\nabla\cdot(\phi(r)\bar{r})=3\phi(r)+r\dfrac{d\phi}{dr}$

Med (11.22) får vi

$$\nabla\cdot(\phi(r)\bar{r})=\underbrace{\nabla\phi(r)}_{=\frac{\partial\phi}{\partial r}\hat{e}_r}\cdot\bar{r}+\phi(r)\underbrace{\nabla\cdot\bar{r}}_{=3}=\frac{\partial\phi}{\partial r}\hat{e}_r\cdot\bar{r}+3\phi(r)=$$

$$=\frac{\partial\phi}{\partial r}\hat{e}_r\cdot(r\hat{e}_r)+3\phi(r)=3\phi(r)+r\frac{d\phi}{dr}$$

(c) $\nabla\cdot((\bar{r}\times\bar{a})\times\bar{r})=-2\bar{a}\cdot\bar{r}$

Vi börjar med uttrycket (11.24):

$$\nabla\cdot((\bar{r}\times\bar{a})\times\bar{r})=\bar{r}\cdot(\nabla\times(\bar{r}\times\bar{a}))-(\bar{r}\times\bar{a})\cdot\underbrace{(\nabla\times\bar{r})}_{=0}=\bar{r}\cdot(\nabla\times(\bar{r}\times\bar{a}))=$$

={Vi fortsätter med (11.25) och använder resultaten i Övningarna 11.3(b) och 10.5(b)}=

$$=\bar{r}\cdot\left(\underbrace{(\bar{a}\cdot\nabla)\bar{r}}_{=\bar{a}}-\bar{a}\underbrace{(\nabla\cdot\bar{r})}_{=3}-\underbrace{(\bar{r}\cdot\nabla)\bar{a}}_{=0}+\bar{r}\underbrace{(\nabla\cdot\bar{a})}_{=0}\right)=\bar{r}\cdot(\bar{a}-3\bar{a})=-2\bar{a}\cdot\bar{r}$$

(d) $\nabla\times(\phi(r)\bar{r})=\bar{0}$

Med (11.23) har vi

$$\nabla\times(\phi(r)\bar{r})=\underbrace{\nabla\phi(r)}_{=\frac{d\phi}{dr}\hat{e}_r}\times\bar{r}+\phi\underbrace{\nabla\times\bar{r}}_{=0}=\frac{d\phi}{dr}\hat{e}_r\times\bar{r}=\frac{d\phi}{dr}\underbrace{\hat{e}_r\times r\hat{e}_r}_{=0}=0$$

PROBLEM 5

Visa att:

$$\nabla\times\left(\bar{r}\times\nabla\phi\right)+\nabla\left(\bar{r}\cdot\nabla\phi\right)=\bar{r}\nabla^2\phi-\nabla\phi$$

Vi måste beräkna $\nabla\times\left(\bar{r}\times\nabla\phi\right)+\nabla\left(\bar{r}\cdot\nabla\phi\right)$.

Vi börjar med den första termen

$$\nabla\times\left(\bar{r}\times\nabla\phi\right)\underbrace{=}_{(12.25)}\underbrace{\left(\nabla\phi\cdot\nabla\right)\bar{r}}_{=\nabla\phi}-\nabla\phi\underbrace{\left(\nabla\cdot\bar{r}\right)}_{=3}-\left(\bar{r}\cdot\nabla\right)\nabla\phi+\bar{r}\underbrace{\left(\nabla\cdot\nabla\phi\right)}_{=\nabla^2\phi}=$$

$$=\nabla\phi-3\nabla\phi-\left(\bar{r}\cdot\nabla\right)\nabla\phi+\bar{r}\nabla^2\phi$$

Med den andra termen, vi får

$$\nabla\left(\bar{r}\cdot\nabla\phi\right)\underbrace{=}_{(12.26)}\underbrace{\left(\nabla\phi\cdot\nabla\right)\bar{r}}_{=\nabla\phi}+\left(\bar{r}\cdot\nabla\right)\nabla\phi+\nabla\phi\times\underbrace{\left(\nabla\times\bar{r}\right)}_{=\bar{0}}+\bar{r}\times\underbrace{\left(\nabla\times\nabla\phi\right)}_{\substack{=\bar{0}\\(12.15)}}=$$

$$=\nabla\phi+\left(\bar{r}\cdot\nabla\right)\nabla\phi$$

Så, uttrycket blir

$$\nabla\times\left(\bar{r}\times\nabla\phi\right)+\nabla\left(\bar{r}\cdot\nabla\phi\right)=\nabla\phi-3\nabla\phi-\left(\bar{r}\cdot\nabla\right)\nabla\phi+\bar{r}\nabla^2\phi+\nabla\phi+\left(\bar{r}\cdot\nabla\right)\nabla\phi=$$

$$=\bar{r}\nabla^2\phi-\nabla\phi$$

PROBLEM 6

Betrakta vektorfältet $\bar{A}=\omega\rho\hat{e}_\varphi$ (i cylidriska koordinater) och sfären S (radie R och centrum i origo).
Visa att

$$\oiint_S \bar{A}\times d\bar{S}=-\frac{8}{3}\pi R^3\omega\hat{e}_z$$

Vi kan använda $\oiint_S d\bar{S}\times\bar{A}=\iiint_V dV\nabla\times\bar{A}$. Så, vi har,

$$\oiint_S \bar{A}\times d\bar{S}=-\oiint_S d\bar{S}\times\bar{A}=-\iiint_V \nabla\times\bar{A}dV$$

Vi kan beräkna rotationen av vektorfältet $\bar{A}=\omega\rho\hat{e}_\varphi$(i ett cylindriskt koordinatsystem) med uttryck (11.45).

$$\nabla\times\bar{A}=\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\varphi}-\frac{\partial A_\varphi}{\partial z}\right)\hat{e}_\rho+\left(\frac{\partial A_\rho}{\partial z}-\frac{\partial A_z}{\partial\rho}\right)\hat{e}_\varphi+\left(\frac{1}{\rho}\frac{\partial\rho A_\varphi}{\partial\rho}-\frac{1}{\rho}\frac{\partial A_\rho}{\partial\varphi}\right)\hat{e}_z$$

$$\nabla\times\left(\omega\rho\hat{e}_\varphi\right)=\left(-\frac{\partial\omega\rho}{\partial z}\right)\hat{e}_\rho+\left(\frac{1}{\rho}\frac{\partial\omega\rho^2}{\partial\rho}\right)\hat{e}_z=2\omega\hat{e}_z$$

Så, integralen blir,

$$\oiint_S \omega\rho\hat{e}_\varphi\times d\bar{S}=-\oiint_S d\bar{S}\times\left(\omega\rho\hat{e}_\varphi\right)=-\iiint_V\nabla\times\left(\omega\rho\hat{e}_\varphi\right)dV=-\iiint_V 2\omega\hat{e}_z dV=$$

$$=-2\omega\hat{e}_z\iiint_V dV=-\frac{8}{3}\pi R^3\omega\hat{e}_z$$

där $\iiint_V dV=V$ är volymen av sfären, $V=\frac{4}{3}\pi R^3$.

PROBLEM 7

Betrakta vektorfältet $\bar{A} = -\omega z\hat{e}_y + \omega y\hat{e}_z$ och kurvan L. L är en ellips som ligger i xy-planet med centrum i origo, radier a och b och som genomlöps i negativ led runt z-axeln.
Visa att:

$$\oint_L \bar{A}\times d\bar{r} = 2\pi ab\omega\,\hat{e}_y$$

Vi kan använda sats: $\oint_L d\bar{r}\times\bar{A} = \iint_S d\bar{S}\times\nabla\times\bar{A}$.

Rotation av vektorfältet $\bar{A} = -\omega z\hat{e}_y + \omega y\hat{e}_z$ kan beräknas med uttryck (10.3). Vi får

$$\nabla\times\bar{A} = 2\omega\,\hat{e}_x$$

Så, integralen blir

$$\oint_L \bar{A}\times d\bar{r} = -\oint_L d\bar{r}\times\bar{A} = -\iint_S d\bar{S}\times\nabla\times\bar{A} = \iint_S \left(\nabla\times\bar{A}\right)\times d\bar{S} = \iint_S 2\omega\,\hat{e}_x\times d\bar{S}$$

L är en ellips som ligger i xy-planet med centrum i origo, med redier a och b och som genomlöps i negativ led rutn z-axeln. Så, normalen är $-\hat{e}_z$ och area av ellipsen är πab. Därför, $d\bar{S} = -\hat{e}_z dS$.
Integralen blir

$$\oint_L \bar{A}\times d\bar{r} = \iint_S 2\omega\,\hat{e}_x\times d\bar{S} = \iint_S 2\omega\,\hat{e}_x\times\left(-\hat{e}_z\right)dS = -\iint_S 2\omega\,\underbrace{\hat{e}_x\times\hat{e}_z}_{=-\hat{e}_y}dS =$$

$$= \iint_S 2\omega\,\hat{e}_y dS = 2\omega\,\hat{e}_y\iint_S dS = 2\pi ab\omega\,\hat{e}_y$$

PROBLEM 8

Elektriska fältet $\bar{E}$ har den skalära potentialen Φ och det magnetiska fältet $\bar{B}$ har vektorpotentialen $\bar{A}$.
Vi kan därmed skriva

$$\bar{E} = \nabla\Phi$$

$$\bar{B} = \nabla\times\bar{A}$$

Påstående (a): $\Phi(\bar{r}) = \Phi_0 = konstant$. Med hjälp av (11.22) fås

$$\nabla\cdot\left(\Phi\bar{B}\right) = \bar{B}\cdot\underbrace{\nabla\Phi}_{=\bar{E}} + \Phi\nabla\cdot\underbrace{\bar{B}}_{=\nabla\times\bar{A}} = \bar{B}\cdot\bar{E} + \Phi\underbrace{\nabla\cdot\nabla\times\bar{A}}_{\substack{=0\\ se\ övning\ 11.2(a)}} = \bar{E}\cdot\bar{B}$$

Med Gauss' sats kan vi skriva

$$\iiint_V \bar{E}\cdot\bar{B}dV = \iiint_V \nabla\cdot\left(\Phi\bar{B}\right)dV \underbrace{=}_{Gauss'sats} \oiint_S \Phi\bar{B}\cdot d\bar{S} \underbrace{=}_{\substack{\Phi=\Phi_0\\ på\ S}} \Phi_0\oiint_S \bar{B}\cdot d\bar{S} \underbrace{=}_{Gauss'sats}$$

$$\underbrace{=}_{Gauss'sats} \Phi_0\iiint_V \nabla\cdot\underbrace{\bar{B}}_{=\nabla\times\bar{A}}dV = \Phi_0\iiint_V \underbrace{\nabla\cdot\nabla\times\bar{A}}_{\substack{=0\\ se\ (11.16)}}dV = 0$$

Påstående (b): $\bar{E}$ är parallell med normalen $\hat{n}$ överallt på S. Vi kan skriva $\bar{E}\times d\bar{S} = \bar{0}$ på ytan S.
Med hjälp av (11.24) fås

$$\nabla\cdot\left(\bar{A}\times\bar{E}\right)=\bar{E}\cdot\left(\nabla\times\bar{A}\right)-\bar{A}\cdot\left(\nabla\times\bar{E}\right) \quad\Rightarrow\quad \bar{E}\cdot\left(\nabla\times\bar{A}\right)=\nabla\cdot\left(\bar{A}\times\bar{E}\right)+\bar{A}\cdot\left(\nabla\times\bar{E}\right)$$

Med Gauss' sats får vi

$$\iiint_V \bar{E}\cdot\underbrace{\bar{B}}_{=\nabla\times\bar{A}}dV=\iiint_V \bar{E}\cdot\left(\nabla\times\bar{A}\right)dV=\iiint_V\left[\nabla\cdot\left(\bar{A}\times\bar{E}\right)+\bar{A}\cdot\left(\nabla\times\bar{E}\right)\right]dV=$$

$$=\oiint_S \underbrace{\left(\bar{A}\times\bar{E}\right)\cdot d\bar{S}}_{\substack{=\bar{A}\cdot\left(\bar{E}\times d\bar{S}\right)\\ trippelprodukter}}+\iiint_V \bar{A}\cdot\underbrace{\left(\nabla\times\nabla\Phi\right)}_{\substack{=0\\ se\,(11.15)}}dV=\oiint_S \bar{A}\cdot\underbrace{\left(\bar{E}\times d\bar{S}\right)}_{=0}=0$$

Påstående (c): $\bar{B}$ är vinkelrät mot $\hat{n}$ överallt på S. Vi kan alltså skriva $\bar{B}\cdot d\bar{S}=0$ på ytan S.

Med hjälp av (11.22) fås

$$\nabla\cdot\left(\Phi\bar{B}\right)=\bar{B}\cdot\underbrace{\nabla\Phi}_{=\bar{E}}+\Phi\nabla\cdot\underbrace{\bar{B}}_{=\nabla\times\bar{A}}=\bar{B}\cdot\bar{E}+\Phi\underbrace{\nabla\cdot\nabla\times\bar{A}}_{\substack{=0\\ se\,(11.16)}}=\bar{E}\cdot\bar{B}$$

Gauss' sats ger

$$\iiint_V \bar{E}\cdot\bar{B}dV=\iiint_V \nabla\cdot\left(\Phi\bar{B}\right)dV\underbrace{=}_{Gauss'sats}\oiint_S \Phi\underbrace{\bar{B}\cdot d\bar{S}}_{=0}=0$$

PROBLEM 9

$$\oiint_S d\bar{S}\times(\bar{v}\times\bar{r})=\frac{8}{3}\pi\bar{v}$$

Sats 15.2: $\oiint_S d\bar{S}\times\bar{v}=\iiint_V(\nabla\times\bar{v})dV$

Så:

$$\oiint_S d\bar{S}\times(\bar{v}\times\bar{r})=\iiint_V\big(\nabla\times(\bar{v}\times\bar{r})\big)dV$$

Med (11.25): $\nabla\times\left(\bar{v}\times\bar{r}\right)=2\bar{v}$

$$\oiint_S d\bar{S}\times(\bar{v}\times\bar{r})=\iiint_V\big(\nabla\times(\bar{v}\times\bar{r})\big)dV=\iiint_V(2\bar{v})dV=2\bar{v}\iiint_V dV=2\bar{v}\,V=\frac{8}{3}\pi\bar{v}$$

Därför volymen av en sfär med radie R=1 är $V=\frac{4}{3}\pi$.