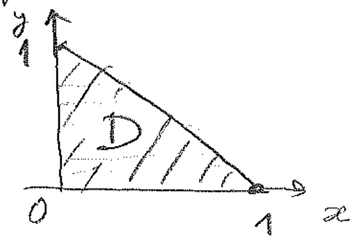
Kap 8.1 Användningar av integraler
Volymberäkningar

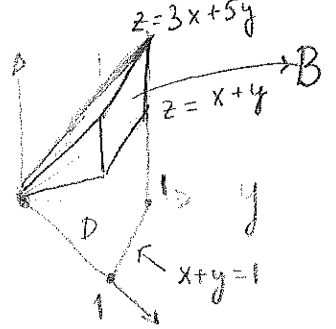
Ex 1 Hitta volymen av regionen mellan planen

$z = x+y$ och $z = 3x+5y$

som ligger ovanför triangeln $D \subset \mathbb{R}^2$ där



Lösning:



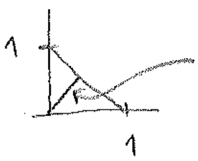
$$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ 0 \le y \le 1-x\}$$

$$\text{Volym} = \iint_D (3x+5y-(x+y))\,dx\,dy =$$

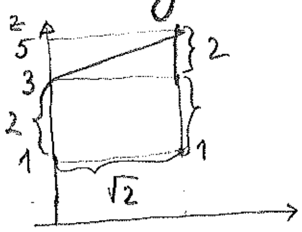
$$= \int_0^1 \int_0^{1-x} (2x+4y)\,dy\,dx = \int_0^1 \left[2xy+2y^2\right]_{y=0}^{1-x} dx =$$

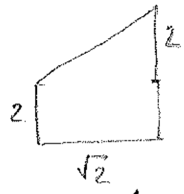
$$= \int_0^1 \left(2x(1-x)+2(1-x)^2\right) dx = \int_0^1 (2-2x)\,dx = \left[2x-x^2\right]_0^1 = 1.$$

Kan också beräknas som $V = \frac{1}{3}\,\text{höjd} \cdot \text{basyta}$ där



$\text{höjd} = \frac{1}{\sqrt{2}}$, $B :=$

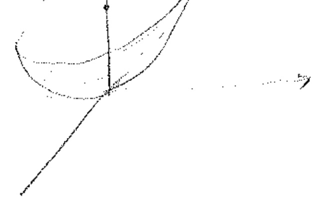




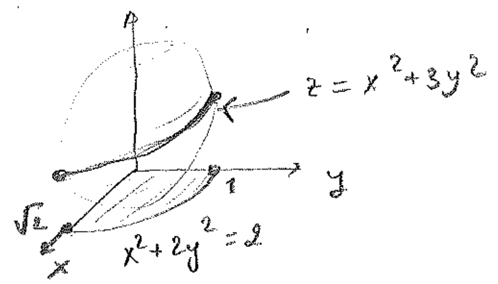
$\text{area} = 3\sqrt{2}.$

Ex 2 Hitta volymen av kroppen K där k är regionen i oktanten $\{x \ge 0,\ y \ge 0,\ z \ge 0\}$ som begränsas uppåt av $z = 4-x^2-y^2$ och nedåt av $z = x^2+3y^2$.

Lösn.



Ytorna skär varandra när
$z = x^2+3y^2 = 4-x^2-y^2$, dvs. när
$\begin{cases} x^2+2y^2 = 2 \quad \text{och} \\ z = x^2+3y^2 \end{cases}$.



Projektionen av k på xy-planet
är $D = \{(x,y) \mid 0 \le y \le 1,\ 0 \le x \le \sqrt{2-2y^2}\}$

Alltså,
$$\text{Volym}(k) = \iint_D \left((4-x^2-y^2)-(x^3+3y^2)\right)dx\,dy =$$

$$\sqrt{2-2y^2} = \sqrt{2}\sqrt{1-y^2}$$

$$= \int_0^1\int_0^{\sqrt{2-2y^2}} (4-2x^2-4y^2)\,dx\,dy = \int_0^1 \Big[\underbrace{4x-\tfrac{2}{3}x^3-4y^2x}_{4x(1-y^2)-\frac{2}{3}x^3}\Big]_{x=0}^{\sqrt{2-2y^2}} dy =$$

$$= \int_0^1 \underbrace{(4\sqrt{2}-\tfrac{2}{3}\cdot 2\sqrt{2})}_{\frac{8}{3}\sqrt{2}}(1-y^2)^{3/2}dy = \begin{Bmatrix} y = \sin\varphi \\ dy = \cos\varphi\, d\varphi \\ \sqrt{1-y^2} = |\cos\varphi| \\ y=0 \Rightarrow \varphi = 0 \\ y=1 \Rightarrow \varphi = \pi/2 \end{Bmatrix} = \frac{8\sqrt{2}}{3}\int_0^{\pi/2}(\cos\varphi)^3\cdot\cos\varphi\, d\varphi =$$

$\cos^2\varphi = \frac{1+\cos 2\varphi}{2}$

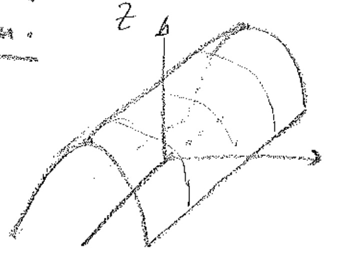
$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi = 2\cos^2\varphi - 1$

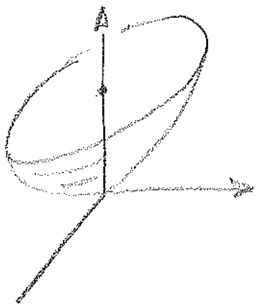
$$= \frac{8\sqrt{2}}{3}\underbrace{\int_0^{\pi/2}\cos^4\varphi\, d\varphi}_{=\frac{3}{8}\cdot\frac{\pi}{2}} = \frac{\pi}{\sqrt{2}}.$$

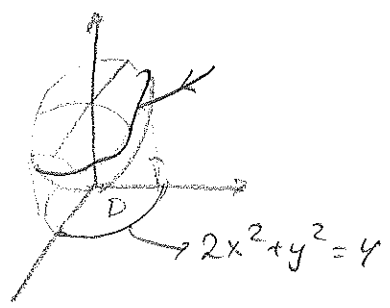
$\cos^4\varphi = (\cos^2\varphi)^2 =$
$\frac{1}{4}(1+\cos 2\varphi)^2 = \frac{1}{4}(1+2\cos 2\varphi+\cos^2 2\varphi)$
$\cos^2 2\varphi = \frac{1+\cos 4\varphi}{2}$. $\quad \frac{1}{4}\cdot\frac{3}{2} = \frac{3}{8}$).

Ex. Bestäm volymen av den kropp k som begränsas av ytorna $z = 8-y^2$ och $z = 4x^2+y^2$

Lösn.







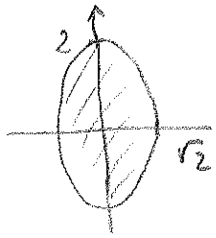
Ytorna skär varandra när $z = 8-y^2 = 4x^2+y^2$, dvs när

$\begin{cases} 2x^2+y^2=4 \\ z = 8-y^2. \end{cases}$

Projektionen D av k på xy-planet ges av
$$D = \{(x,y) \mid 2x^2+y^2 \le 4\}.$$

Vi får

$$\text{Volym}(K) = \iint_D \left((8-y^2) - (4x^2+y^2)\right) dx\,dy = 2\iint_{\substack{D \\ \{2x^2+y^2 \le 4\}}} (4 - 2x^2 - y^2)\,dx\,dy$$



$$= \left\{ \begin{array}{l} \tilde{x} = \sqrt{2}\,x \\ x = \frac{\tilde{x}}{\sqrt{2}} \\ dx = \frac{d\tilde{x}}{\sqrt{2}} \\ D \rightsquigarrow \{\tilde{x}^2 + y^2 \le 4\} \end{array} \right\} = \sqrt{2} \iint_{\tilde{x}^2+y^2 \le 4} (4 - \tilde{x}^2 - y^2)\,d\tilde{x}\,dy =$$

$$= \left\{ \begin{array}{l} \tilde{x} = r\cos\varphi \\ y = r\sin\varphi \end{array} \right\} = \sqrt{2} \int_0^{2\pi} \left(\int_0^2 \underbrace{(4 - r^2)\,r}_{2r - r^3}\,dr \right) d\varphi = \sqrt{2}\cdot 2\pi \cdot \left(2r^2 - \frac{r^4}{4}\right)_0^2 =$$

$$2\pi\sqrt{2}\,(8-4) = \underline{8\pi\sqrt{2}}.$$