Kap. 10.3 Stokes sats

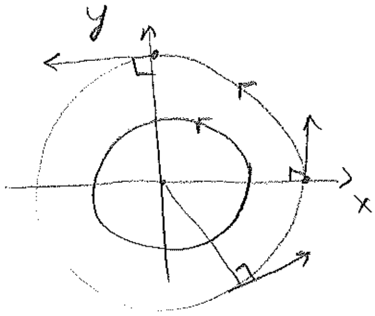
Def Rotationen av det 3-dim. vektorfältet $\vec{u} = (u_1, u_2, u_3)$ av tre variabler $\vec{x} = (x_1, x_2, x_3)$ är vektorfältet

$$\operatorname{rot}\vec{u} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{bmatrix} =$$

$$= \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \ \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right)$$

Ibland kallas $\operatorname{rot}\vec{u}$ för $\nabla \times \vec{u}$ eller $\operatorname{curl}\vec{u}$.

Ex 1 $\vec{u}(x,y,z) = (-y, x, 0)$



hastighetsfält $\vec{u}$ för rotationen. OBS: $\vec{u}(x,y,z) \perp (x,y,z)$ $\forall (x,y,z)$ samt $\vec{u}(x,y,z) \perp (0,0,1)$. Alltså, $\vec{u} \parallel \underbrace{(0,0,1)}_{\text{rot. axeln}} \times \underbrace{(x,y,z)}_{\vec{x}}$.

$$\operatorname{rot}\vec{u} = (0, 0, 1-(-1)) = (0,0,2).$$

Mer allmänt, om $\vec{u} = \vec{\omega} \times \vec{x}$ är hastighetsvektor för rotationen kring en axel med riktning $\vec{\omega} \in \mathbb{R}^3$ och vinkelhastigheten $|\vec{\omega}|$, så är

$\operatorname{rot}\vec{u} = 2\vec{\omega}$. (se ex. 11 sid. 377).

Ex 2 Magnetfältet $B = \frac{(-y, x, 0)}{x^2+y^2}$, $\boxed{x^2+y^2 \neq 0}$ runt den strömgenomflytna z-axeln är virvelfritt (dvs $\operatorname{rot} B = 0$)

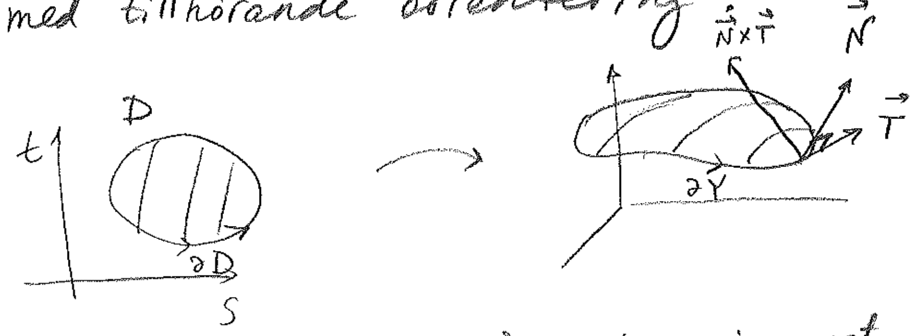
$$\operatorname{rot} B = \begin{vmatrix} e_x & e_y & e_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix} = \left(0, \ 0, \ \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right) =$$

$$= \frac{x^2+y^2 - x \cdot 2x + x^2+y^2 - 2y^2}{(x^2+y^2)^2} = 0.$$

Stokes' sats

Låt $Y \subset \mathbb{R}^3$ vara en C^1-yta given av $\vec{r}(s,t)$, $(s,t) \in D$, där $D \subset \mathbb{R}^2$ är ett kompakt område med C^1 rand ∂D.

Def Randen ∂Y av Y är bilden av ∂D under $\vec{r}$ med tillhörande orientering



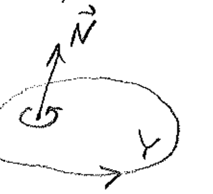
$\vec{T}$ — tangent till ∂Y
$\vec{N}$ — normal till Y

Notera att $\vec{N} \times \vec{T}$ pekar in mot ytan.

Sats (Stokes sats)

Om $\vec{u} = (u_1, u_2, u_3)$ är ett C^1-fält definierat i en öppen mängd $\Omega \subset \mathbb{R}^3$, och $Y \subset \Omega$, så gäller

$$\int_{\partial Y} u \cdot dr = \iint_Y (\text{rot}\, \vec{u}) \cdot d\vec{S} = \iint (\text{rot}\, \vec{u}) \cdot \vec{N}\, dS$$



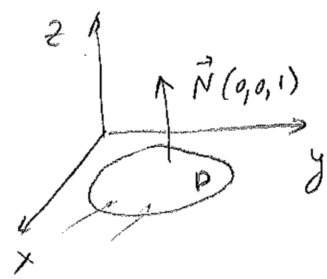
cirkulationen längs randen av Y

"Summa av rotationer över ytan"
"flödet av vektorn rot $\vec{u}$"

Kommentar Detta är en generalisering av Greens formel till 3 dim.:

Ex Låt $\vec{u} = (P(x,y), Q(x,y), 0)$

och $Y = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in D, z = 0\}$



$$\int_{\partial Y} \vec{u} \cdot d\vec{r} \underset{\text{Stokes}}{=} \underbrace{\iint_Y (\text{rot}\, \vec{u}) \cdot \vec{N}\, dS}$$

$$\| \qquad\qquad \|$$

$$\int_{\partial D} P\,dx + Q\,dy \qquad \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dS$$

Ex $\boxed{\frac{1}{\pi \varepsilon^2} \iint_{C_\varepsilon} (\text{rot}\, \vec{u}) \cdot \vec{N}\, dS} \xrightarrow{\varepsilon \to 0} \text{rot}\, \vec{u}(\vec{y}) \cdot \vec{N}\, dS.$

Skivans area ↗ ($\pi\varepsilon^2$)

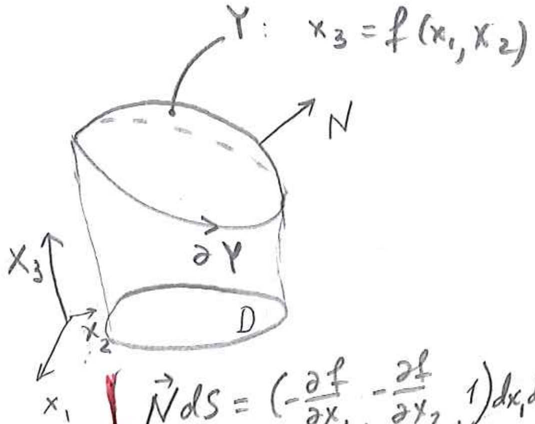
C_ε: skiva kring $\vec{y}$, rad ε.

Stokes sats: $\frac{1}{\pi \varepsilon^2} \int_{\partial C_\varepsilon} u \cdot dr \longrightarrow \text{rot}\, \vec{u}(\vec{y}) \cdot \vec{N}$

"cirkulation på en liten cirkel kring y vinkelrätt med $\vec{N}$ är rotation."

Bevis av Stokes sats (i fallet dä Y är en funktionsyta med ekv. $x_3 = f(x_1, x_2)$ $(x_1, x_2) \in D$, och $\vec{u} \in C^2$.

$$\int_{\partial Y} \vec{u} \cdot d\vec{r} = \int_{\partial Y} u_1 dx_1 + u_2 dx_2 + u_3 dx_3 =$$



$\vec{N} dS = \left(-\frac{\partial f}{\partial x_1}, -\frac{\partial f}{\partial x_2}, 1\right) dx_1 dx_2$
se ex 3 sid 361

Man kan tänka så:

$\vec{r} = x_1(t), x_2(t), f(x_1(t), x_2(t))$

$\vec{r}'(t) = \left(x_1'(t), x_2'(t), \frac{\partial f}{\partial x_1} x_1'(t) + \frac{\partial f}{\partial x_2} \cdot x_2'(t)\right)$

$\vec{u} \cdot d\vec{r} = \left(u_1 x_1' + u_2 x_2' + u_3 \left(\frac{\partial f}{\partial x_1} x_1' + \frac{\partial f}{\partial x_2} x_2'\right)\right) dt =$

$= u_1 dx_1 + u_2 dx_2 + u_3 \frac{\partial f}{\partial x_1} dx_1 + u_3 \frac{\partial f}{\partial x_2} dx_2$ där $dx_1 = x_1' dt$, $dx_2 = x_2' dt$.

$$= \int_{\partial D} u_1 dx_1 + u_2 dx_2 + u_3 \left(\frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2\right) =$$

$$= \int_{\partial D} \underbrace{\left(u_1 + u_3 \frac{\partial f}{\partial x_1}\right)}_{P} dx_1 + \underbrace{\left(u_2 + u_3 \frac{\partial f}{\partial x_2}\right)}_{Q} dx_2 \overset{\text{Greens}}{\underset{\text{sats}}{=}}$$

$$= \iint_D \left[\frac{\partial}{\partial x_1}\left(u_2 + u_3 \frac{\partial f}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(u_1 + u_3 \frac{\partial f}{\partial x_1}\right)\right] dx_1 dx_2 =$$

$u_2 = u_2(x_1, x_2, f(x_1, x_2))$

$$= \iint_D \left[\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_2}{\partial x_3} \frac{\partial f}{\partial x_1}\right) + \left(\frac{\partial u_3}{\partial x_1} + \cancel{\frac{\partial u_3}{\partial x_3} \frac{\partial f}{\partial x_1}}\right) \cdot \frac{\partial f}{\partial x_2} + \cancel{u_3 \frac{\partial^2 f}{\partial x_1 \partial x_2}}\right.$$

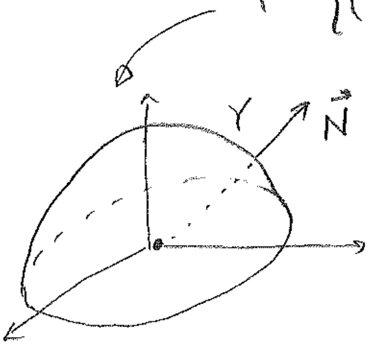
$$\left. - \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_1}{\partial x_3} \frac{\partial f}{\partial x_2}\right) - \left(\frac{\partial u_3}{\partial x_2} + \cancel{\frac{\partial u_3}{\partial x_3} \frac{\partial f}{\partial x_2}}\right) \frac{\partial f}{\partial x_1} - \cancel{u_3 \frac{\partial^2 f}{\partial x_1 \partial x_2}}\right] dx_1 dx_2 =$$

$$= \iint_D - \frac{\partial f}{\partial x_1}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) - \frac{\partial f}{\partial x_2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) + \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) dx_1 dx_2$$

$$= \iint_D \operatorname{rot} \vec{u} \cdot \vec{N} \, dS$$

$\underline{Ex\ 3}$ $\quad \vec{u} = (-y,\ 2x,\ x+z)$

$Y = \{(x,y,z) \in R^3 \mid x^2+y^2+z^2 = 1, \quad z \geq 0\}$.



Verifiera Stokes sats:

$$\begin{matrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_1 & u_2 & u_3 \end{matrix}$$

1) $$\text{rot}\,\vec{u} = \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z},\ \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x},\ \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right) =$$

$$= (0-0,\ 0-1,\ 2-(-1)) = (0,\ -1,\ 3).$$

Parametriserar Y: $\vec{r}(\theta,\varphi) = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$

$0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \varphi \leq 2\pi$

$$d\vec{S} = \vec{N}\cdot dS = \vec{N}\,|r'_\theta \times r'_\varphi|\,d\theta d\varphi = \frac{\vec{r}}{1}\cdot \sin\theta\, d\theta d\varphi$$

enhetsnorm. $\qquad$ sid 307

Alltså, $$\iint_Y \text{rot}\,\vec{u} \cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi/2} (0,-1,3)\cdot(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)\sin\theta\, d\theta d\varphi$$

$$= \int_0^{2\pi}\int_0^{\pi/2} (-\sin^2\theta\sin\varphi + 3\cos\theta\sin\theta)\,d\theta\, d\varphi =$$

$$\underbrace{\int_0^{2\pi}\sin\varphi\, d\varphi}_{=0} \cdot \int_0^{\pi/2}(-\sin^2\theta)\,d\theta + 2\pi\cdot\frac{3}{2}\Big[\sin^2\theta\Big]_0^{\pi/2} = 3\pi.$$

2) Parametriserar ∂Y: $\vec{r}(t) = (\cos t,\ \sin t,\ 0), \quad 0 \leq t \leq 2\pi$,

$$\oint_{\partial Y} \vec{u}\cdot d\vec{r} = \int_0^{2\pi} \underbrace{(-\sin t,\ 2\cos t,\ \ldots)}_{u(\vec{r}(t))} \cdot \underbrace{(-\sin t,\ \cos t,\ 0)}_{\vec{r}\,'}\,dt =$$

$$= \int_0^{2\pi} (\sin^2 t + 2\cos^2 t)\,dt = \int_0^{2\pi}(1 + \underbrace{\cos^2 t}_{=\frac{\cos 2t + 1}{2}})\,dt = 2\pi\cdot\frac{3}{2} = 3\pi.$$

0