Kap. 2.7 Differentialer

Låt $f: D \to \mathbb{R}$, $D \subset \mathbb{R}^n$ öppen, f differentierbar.

Def Den *linjära funktionen* $df(\vec{x}): \mathbb{R}^n \to \mathbb{R}$

$$df(\vec{x})(\vec{h}) = f'_{x_1}(\vec{x})\, h_1 + \dots + f'_{x_n}(\vec{x})\, h_n = \left(f'_{x_1}(\vec{x}), \dots f'_{x_n}(\vec{x})\right) \cdot \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}.$$

kallas för *differentialen* av f i punkten $\vec{x}$.

Definitionen av differentierbarhet kan skrivas:

$$f(\vec{x}+\vec{h}) - f(\vec{x}) = df(\vec{x})(\vec{h}) + |\vec{h}|\, \rho(\vec{x}, \vec{h}) \quad \text{där} \quad \lim_{\vec{h} \to 0} \rho(\vec{x}, \vec{h}) = 0 .$$

Ex $df(\vec{x}): \mathbb{R}^n \to \mathbb{R}$ representeras av matrisen $\left(f'_{x_1}(\vec{x}), \dots f'_{x_n}(\vec{x})\right)$

Ex Om $f(\vec{x}) = x_1$, får vi

$$dx_1 := df(\vec{x}) = \left(\frac{\partial x_1}{\partial x_1}, \frac{\partial x_1}{\partial x_2} \dots \frac{\partial x_1}{\partial x_n}\right) = (1, 0 \dots 0).$$

dvs $dx_1(\vec{h}) = h_1$

På samma sätt, $dx_j(\vec{h}) = h_j \quad j = 1, \dots n$.

Ex $df(h) = f'_{x_1}(x)\, h_1 + \dots f'_{x_n}(x)\, h_n = f'_{x_1}(x) \cdot dx_1(h) + \dots f'_{x_n}(x)\, dx_n(h)$

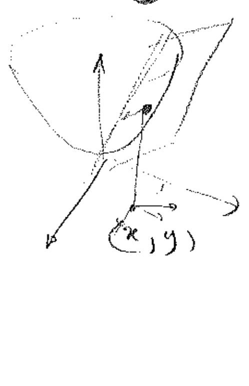
Alltså, $\boxed{df = f'_{x_1}(x)\, dx_1 + \dots f'_{x_n}\, dx_n.}$ för alla $h \in \mathbb{R}^n$.

Ex $f(x,y) = y^3 \ln x, \quad x > 0$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \frac{y^3}{x} dx + 3y^2 \ln x\, dy$$

Ex $f(x,y) = x^2 + y^2$

$$df = 2x\, dx + 2y\, dy .$$



Räkneregler:
- $d(f+g) = df + dg$
- $d(f \cdot g) = g\, df + f \cdot dg$
- $d\left(\frac{f}{g}\right) = \frac{g\, df - f\, dg}{g^2}$
- $d(f \circ g)(\vec{x}) = df(g(\vec{x})) \cdot dg(\vec{x})$

$\underline{Ex}$ $f(x,y) = e^{xy^2} + \sin(xy)$

$$df = d(e^{xy^2}) + d\sin(xy) = e^{xy^2} \cdot d(xy^2) + \cos(xy) \cdot dxy =$$

$$= e^{xy^2}\left(\underline{y^2 \cdot dx} + x\underbrace{dy^2}_{2xydy}\right) + \cos(xy) \cdot (xdy + \underline{ydx}) =$$

$$= \left(\underbrace{y^2 e^{xy^2} + y\cos(xy)}_{\frac{\partial f}{\partial x}}\right)dx + \left(\underbrace{2xye^{xy^2} + x\cos(xy)}_{\frac{\partial f}{\partial y}}\right)dy$$