K. 2.5 Partiella derivator av högre ordning

Ex 1 $f(x,y) = x^3 \cos 2y$

Första ordningens derivator

$$\frac{\partial f}{\partial x} = 3x^2 \cos 2y \qquad \frac{\partial f}{\partial y} = -2x^3 \sin 2y$$

Andra ordn. derivator:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 6x \cos 2y, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -6x^2 \sin 2y$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -6x^2 \sin 2y, \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -4x^3 \cos 2y$$

Tredje ordn.

$$\frac{\partial^3 f}{\partial x^3} = 6 \cos 2y \quad \text{osv.}$$

OBS: $\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}}$

Def Låt $f: D \to \mathbb{R}$, $D \subset \mathbb{R}^n$ öppen. Vi säger att f är av klass C^k ($f \in C^k(D)$) om alla part. deriv. t.o.m ordning k existerar och är kontinuerliga i D.

Sats För varje funktion $f(\vec{x}) = f(x_1, .. x_n)$ av klass C^2 gäller att $f''_{jk} = f''_{kj}$ $\left(\text{dvs } \frac{\partial^2 f}{\partial x_k \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_k}\right)$, $j, k = 1, .. n$.

Bevis — läs själva! Ideer liknar beviset för sats 3 sid. 56 att $f \in C^1 \Rightarrow f$ är differentierbar.

Ex Bestäm alla C^2-lösningar $f(x,t)$ till vågekvationen

$$(*) \quad \frac{\partial^2 f}{\partial t^2} - c^2 \frac{\partial^2 f}{\partial x^2} = 0 \qquad \left(\Longleftrightarrow \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) f = 0 \right.$$

$$\left. = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial f}{\partial t} + c\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial t^2} - \cancel{c \cdot \frac{\partial^2 f}{\partial x \partial t}} + \cancel{c\frac{\partial^2 f}{\partial t \partial x}} - c^2 \frac{\partial^2 f}{\partial x^2} \right)$$

m.h.a. variabelbytet $\begin{cases} u = x + ct \\ v = x - ct. \end{cases}$

Lösn. Kedjeregeln ger:

$$\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial f}{\partial v}\cdot\dfrac{\partial v}{\partial x} = \dfrac{\partial f}{\partial u} + \dfrac{\partial f}{\partial v} \\ \dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial t} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial t} = c\left(\dfrac{\partial f}{\partial u} - \dfrac{\partial f}{\partial v}\right) \end{cases} \quad \Longleftrightarrow$$

$$\begin{cases} \dfrac{\partial}{\partial t} + c\dfrac{\partial}{\partial x} = 2c\dfrac{\partial}{\partial u} \\ \dfrac{\partial}{\partial t} - c\dfrac{\partial}{\partial x} = -2c\dfrac{\partial}{\partial v}. \end{cases}$$

Vi ser att f uppfyller $(*)$ $\Longleftrightarrow$ $\underbrace{-2c\frac{\partial}{\partial v}\left(2c\frac{\partial}{\partial u} f\right) = -4c^2 \frac{\partial^2 f}{\partial u \partial v} = 0}_{\Longleftrightarrow \boxed{\frac{\partial}{\partial u}\left(\frac{\partial f(u,v)}{\partial v}\right) = 0}}$ $(c \neq 0)$.

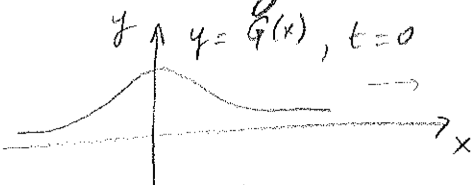
Vi integrerar:

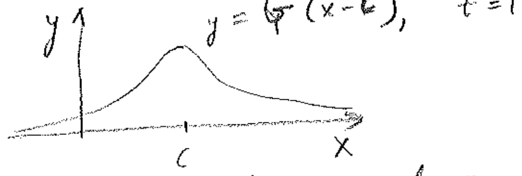
$\frac{\partial f(u,v)}{\partial v} = g(v)$, där $g(v)$ är någon funktion.

$$f(u,v) = \int^{v} g(x)\,dx + H(u) = G(v) + H(u)$$

Vi har alltså $f(x,t) = G(x-ct) + H(x+ct)$.

OBS $G(x-ct)$ är en resande våg som åker med hastighet c åt höger:





$H(x+ct)$ är en resande våg åt vänster.

Uppgift 2.59.

Bestäm allmänna lösn. till

$$x \cdot \frac{\partial^2 f}{\partial x^2} - y \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial x} = 0 \qquad (**)$$

genom att införa variablerna

$$\begin{cases} u = y \\ v = xy \end{cases}, \quad y \neq 0.$$

Lösn.

$$\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial x} = 0 + \dfrac{\partial f}{\partial v} \cdot y = u \cdot \dfrac{\partial f}{\partial v}. \\[2ex] \dfrac{\partial f}{\partial y} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial y} = \dfrac{\partial f}{\partial u} \cdot 1 + \dfrac{\partial f}{\partial v} \cdot \underbrace{x}_{\frac{v}{y} = \frac{v}{u}} = \dfrac{\partial f}{\partial u} + \dfrac{v}{u}\dfrac{\partial f}{\partial v}, \end{cases}$$

Alltså,

$$\begin{cases} \dfrac{\partial}{\partial x} = u \cdot \dfrac{\partial}{\partial v} \\[2ex] \dfrac{\partial}{\partial y} = \left(\dfrac{\partial}{\partial u} + \dfrac{v}{u} \cdot \dfrac{\partial}{\partial v}\right); \end{cases} \qquad \begin{array}{l} y = u, \quad u \neq 0 \\ x = \frac{v}{u}. \end{array}$$

(**) Kan skrivas om:

$$\frac{v}{u} \cdot u \frac{\partial}{\partial v}\left(u \frac{\partial f}{\partial v}\right) - u \cdot u \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial u} + \frac{v}{u}\frac{\partial f}{\partial v}\right) + u \frac{\partial f}{\partial v} = 0$$

$$\cancel{vu \cdot \frac{\partial^2 f}{\partial v^2}} - u^2\left(\frac{\partial^2 f}{\partial u \partial v} + \cancel{\frac{1}{u}\frac{\partial f}{\partial v}} + \cancel{\frac{v}{u}\frac{\partial^2 f}{\partial v^2}}\right) + \cancel{u\frac{\partial f}{\partial v}} = 0$$

$$-\underbrace{u^2}_{\neq 0} \frac{\partial^2 f}{\partial u \partial v} = 0 \quad \underset{\text{för } u \neq 0}{\Longleftrightarrow} \quad \frac{\partial^2 f}{\partial u \partial v} = 0$$

$$f(u, v) = G(u) + H(v) \qquad \text{(som i ex ovan)}$$

$$f(x, y) = G(y) + H(xy),$$ där G och H är godtyckliga C^2-funktioner.