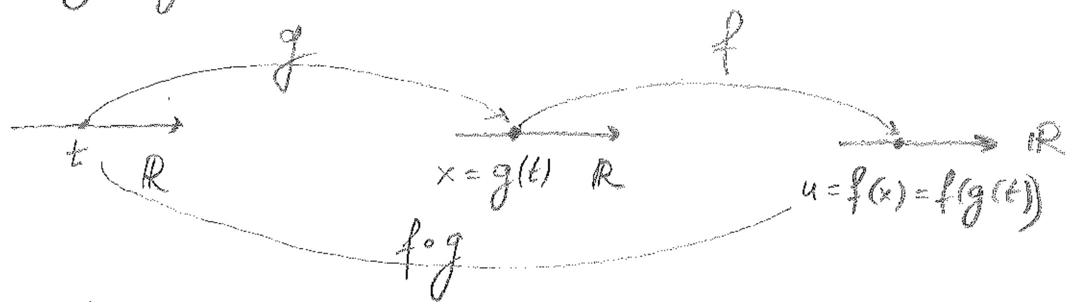
K. 2.3 Kedjeregeln

- I 1 dim.



$(f\circ g)'(t) = f'(g(t))\cdot g'(t).$

Med $x = g(t)$ och $u = f(x)$ kan detta skrivas:

$$\frac{du}{dt} = \frac{du}{dx}\cdot\frac{dx}{dt}.$$

- Om $g = g(t_1, \dots t_n) : \mathbb{R}^n \to \mathbb{R}$, $f : \mathbb{R}\to\mathbb{R}$, så

$f(g(t)) = f(g(t_1, \dots t_n))$, och

$\frac{\partial}{\partial t_j} f(g(t)) = f'(g(t))\cdot \frac{\partial g}{\partial t_j}(t) \qquad j = 1, \dots n.$

Med $x = g(\underset{\mathbb{R}^n}{\underset{\cap}{t}})$, $u = f(x)$,

$$\frac{\partial u}{\partial t_j} = \frac{du}{dx}\cdot\frac{\partial x}{\partial t_j}.$$

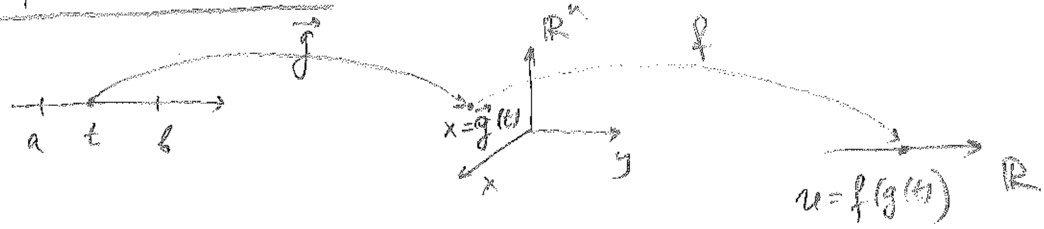
Ex $\quad \frac{\partial}{\partial t}\sin(s+2t) = \cos(s+2t)\cdot\frac{\partial}{\partial t}(s+2t) = 2\cos(s+2t)$

Ex $\quad r = \sqrt{x^2+y^2+z^2}$

$f = f(r) = f(r(x,y,z)).$

$$\frac{\partial f}{\partial x} = f'(r(x,y,z))\cdot\frac{\partial r}{\partial x} = f'(r)\cdot\frac{2x}{2\sqrt{x^2+y^2+z^2}} = f'(r)\cdot\frac{x}{r}$$

I flera variabler



Sats 4 Låt $f(x) = f(x_1, \ldots x_n)$ vara en differentierbar funktion av n variabler, och antag att $g_1(t), \ldots g_n(t)$ är deriverbara på interv. $(a,b) \subset \mathbb{R}$.
Då är den sammansatta funktionen $f(\vec{g}(t))$ också deriverbar på (a,b), och

$$\frac{d}{dt} f(g(t)) = f'_{x_1}(\vec{g}(t)) \cdot g_1'(t) + \ldots f'_{x_n}(\vec{g}(t)) \cdot g_n'(t).$$

Med $\vec{x} = \vec{g}(t)$, $u = f(x)$, har vi

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1} \cdot \frac{dx_1}{dt} + \ldots \frac{\partial u}{\partial x_n} \cdot \frac{dx_n}{dt} = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j} \frac{dx_j}{dt}$$

Ex $f(\vec{x}) = f(x,y) = xy^2$,
$\vec{x}(t) = (x(t), y(t)) = (\sin t, e^{2t})$;
$(f(\vec{x}(t)) = \sin t \cdot e^{4t})$. Verifiera att kedjeregeln gäller.

Vi behöver $f'_x(x,y) = y^2$,
$f'_y(x,y) = 2xy$.

Kedjeregeln ger:

$$\frac{d}{dt} f(\vec{x}(t)) = f'_x(\vec{x}(t)) \cdot x'(t) + f'_y(\vec{x}(t)) \cdot y'(t) = y^2 \cdot \cos t + 2xy \cdot 2e^{2t}$$

$$= e^{4t} \cdot \cos t + 2e^{2t} \cdot \sin t \cdot 2e^{2t} = e^{4t}(\cos t + 4\sin t)$$

$\|$

Deriverar direkt:

$$f(\vec{x}(t)) = (\sin t \cdot e^{4t})' = \cos t\, e^{4t} + 4 \sin t\, e^{4t} = e^{4t}(\cos t + 4\sin t)$$

stämmer.

Bevis Då f är differentierbar, gäller:

$$(*)\quad f(\vec{x}+\vec{h}) - f(\vec{x}) = \sum_{j=1}^{n} f'_{x_j}(\vec{x}) \cdot h_j + |\vec{h}| \cdot \rho(\vec{h})$$

där $\lim_{h \to 0} \rho(\vec{h}) = 0$

Med $\vec{x} = \vec{g}(t)$ och $\vec{h} = \vec{g}(t+k) - \vec{g}(t)$ har vi

$$\vec{x} + \vec{h} = \vec{g}(t+k)$$

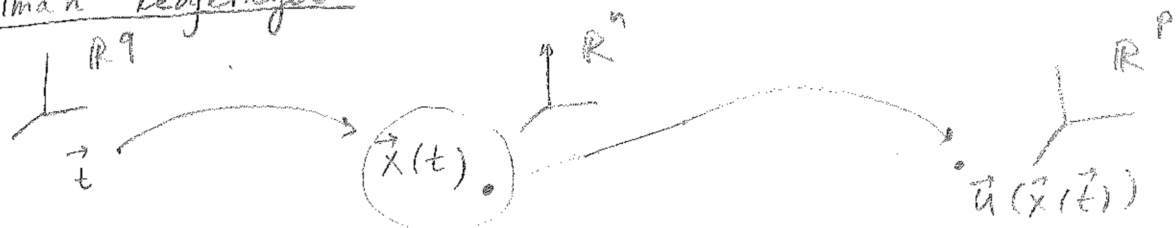
$$\circledast \quad \frac{f(\vec{g}(t+k)) - f(g(t))}{k} = \sum f'_{x_j}(\vec{g}(t)) \cdot \frac{(g_j(t+k) - g_j(t))}{k}$$

$$+ \frac{|\vec{g}(t+k) - \vec{g}(t)|}{k} \cdot \rho\left(g(t+k) - g(t)\right)$$

$$\to \sum f'_{x_j}(\vec{g}(t)) \cdot g'_j(t)$$

$$\text{ty} \begin{cases} \dfrac{\vec{g}(t+k) - \vec{g}(t)}{k} \to \vec{g}'(t) \\ \rho(g(t+k) - g(t)) \to 0 \qquad \text{då } k \to 0 \end{cases}$$

<u>Allmän kedjeregel</u>

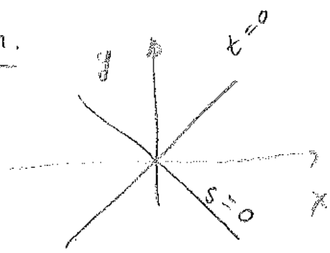


$$\frac{\partial u_i}{\partial t_k} = \sum_{j=1}^{n} \frac{\partial u_i}{\partial x_j} \frac{\partial x_j}{\partial t_k}, \qquad \begin{matrix} i = 1, \dots p \\ k = 1, \dots q \end{matrix}$$

<u>Ex</u> Finn alla C^1-funktioner som uppfyller

$$\frac{\partial f}{\partial x} = -\frac{\partial f}{\partial y}, \qquad (x,y) \in \mathbb{R}^2,$$ genom att introducera de nya variablerna $\begin{cases} s = x+y \\ t = x-y \end{cases}$.

<u>Lösn.</u>



Skriv $f = f(s,t) = f(s(x,y), t(x,y))$

$$\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial s} \cdot \dfrac{\partial s}{\partial x} + \dfrac{\partial f}{\partial t} \dfrac{\partial t}{\partial x} = \dfrac{\partial f}{\partial s} + \dfrac{\partial f}{\partial t} \\ \dfrac{\partial f}{\partial y} = \dfrac{\partial f}{\partial s} \dfrac{\partial s}{\partial y} + \dfrac{\partial f}{\partial t} \dfrac{\partial t}{\partial y} = \dfrac{\partial f}{\partial s} - \dfrac{\partial f}{\partial t}. \end{cases}$$

ekv. ger: $\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} = 0 \iff 2\dfrac{\partial f}{\partial s} = 0$. Alltså,

$$f(s,t) = \varphi(t) = \varphi(x-y)$$

där φ är en godtycklig funktion.