

SF1661 HT20

Öreläsning 10

23/9

Komplexa tal

$$\mathbb{C} = \{z = x + iy \ ; \ x, y \in \mathbb{R}, \ i^2 = -1\}$$

$$z + w, \ z - w, \ z \cdot w, \ \frac{z}{w}$$

$$|z|, \ \operatorname{Re}(z), \ \operatorname{Im}(z), \ \overline{z}$$

$$|z + w| \leq |z| + |w|$$

$$z = |z| (\cos \theta + i \sin \theta) = |z| e^{i\theta}$$

Ekvationslösning och faktorisering
av polynom

Varför komplexa tal?

• Historia

• Matematiska skäl

• Tillämpningar

Historia Lösning av 3:e grads ekv.

Cardanos formel (1500-tal):

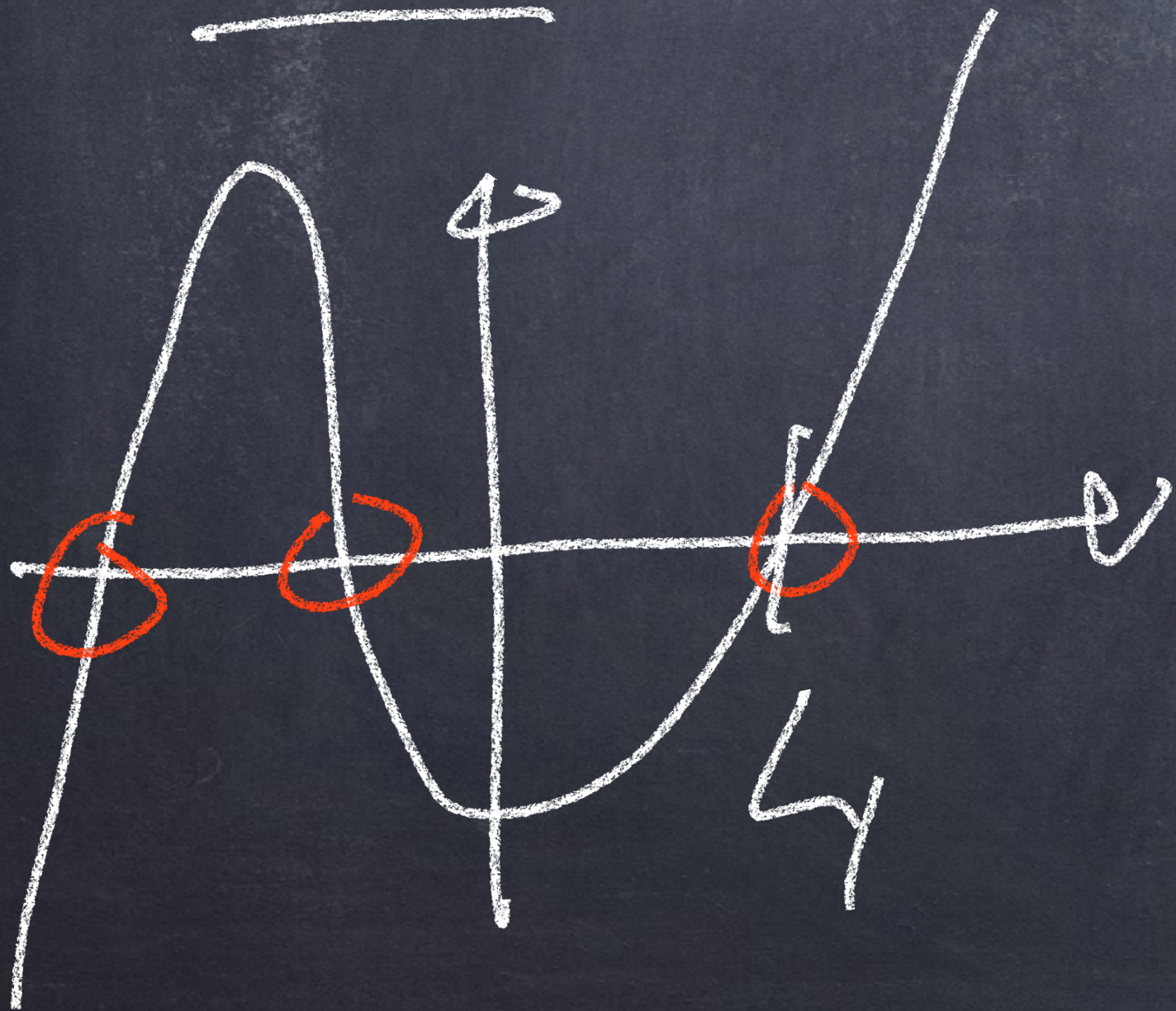
$$X^3 + Px + Q = 0 \quad \text{har lösningar}$$

$$X = \sqrt[3]{-\frac{Q}{2} + \sqrt{\left(\frac{Q}{2}\right)^2 + \left(\frac{P}{3}\right)^3}} + \sqrt[3]{-\frac{Q}{2} - \sqrt{\left(\frac{Q}{2}\right)^2 + \left(\frac{P}{3}\right)^3}}$$

behöver ni
inte kunna :-)

Även vid ekvationen med reella koefficienter
och reella lösningar ger Cardanos formel
räkning med "imaginära tal" på vägen."

Ex: $x^3 - 15x - 4 = 0$ | $p = -15$, $q = -4$



$$\left(\frac{q}{2}\right)^2 = 4 \quad \left(\frac{p}{3}\right)^3 = -125$$

$$x = \sqrt[3]{2 + \sqrt{4 - 125}} + \sqrt[3]{2 - \sqrt{4 - 125}}$$
$$= \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i} = 4$$

Matematiska skäl

$$\bullet P(z) = a_n z^n + \dots + a_1 z + a_0 = 0, \quad a_i \in \mathbb{C}$$
$$\iff$$

$$P(z) = (z - \alpha_1)(z - \alpha_2) \dots (z - \alpha_n), \quad \alpha_i \in \mathbb{C}$$

$$\bullet e^{i\pi} = -1$$

"Kortaste vägen mellan två utsagor"

- om reella tal går ofta genom komplexa talplanet

Tillämpningar

• "jw-metoden" växelströmskretsar

• Kvantmekanikens grundekvation

("Schrödinger equation")


$$i \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x) \psi$$

$$\mathbb{C} = \{a + ib; a, b \in \mathbb{R}\}$$

• $+$, $-$, $\cdot$, $/$

med tillägg

def. "som vanligt"

$$i^2 = -1$$

$$(a + ib) + (c + id) = (a + c) + (b + d)i$$

$$(a + ib)(c + id) = ac + iad + ibc + i^2 bd$$

$$= (ac - bd) + i(ad + bc)$$

$$\frac{a+ib}{c+id} \stackrel{||}{=} \frac{(a+ib)(c-id)}{(c+id)(c-id)} \stackrel{||}{=} \frac{(ac+bd) + i(bc-ad)}{c^2 - (id)^2} \stackrel{||}{=}$$

$$\stackrel{||}{=} \frac{ac+bd}{c^2+d^2} + i \frac{bc-ad}{c^2+d^2}$$

$(c+id \neq 0)$

$$\text{Låt } Z = \left(\frac{7}{2} + i\right), \quad W = 1 - 2i$$

Beräkna

$$Z + W = \left(\frac{7}{2} + i\right) + (1 - 2i) = \left(\frac{7}{2} + 1\right) + (1 - 2)i$$

$$2Z - W = 2\left(\frac{7}{2} + i\right) - (1 - 2i) = 7 + 2i - 1 + 2i$$

$$Z \cdot W = \left(\frac{7}{2} + i\right)(1 - 2i) = \left(\frac{7}{2} \cdot 1 - i \cdot 2i\right) + i\left(1 - \frac{7}{2}\right)$$

$$\frac{Z}{W} = \frac{\left(\frac{7}{2} + i\right)}{1 - 2i} = \frac{\left(\frac{7}{2} + i\right)(1 + 2i)}{(1 - 2i)(1 + 2i)} = \frac{\dots}{1 + 2^2}$$

$$\text{Låt } Z = \left(\frac{7}{2} + i \right), \quad W = 1 - 2i$$

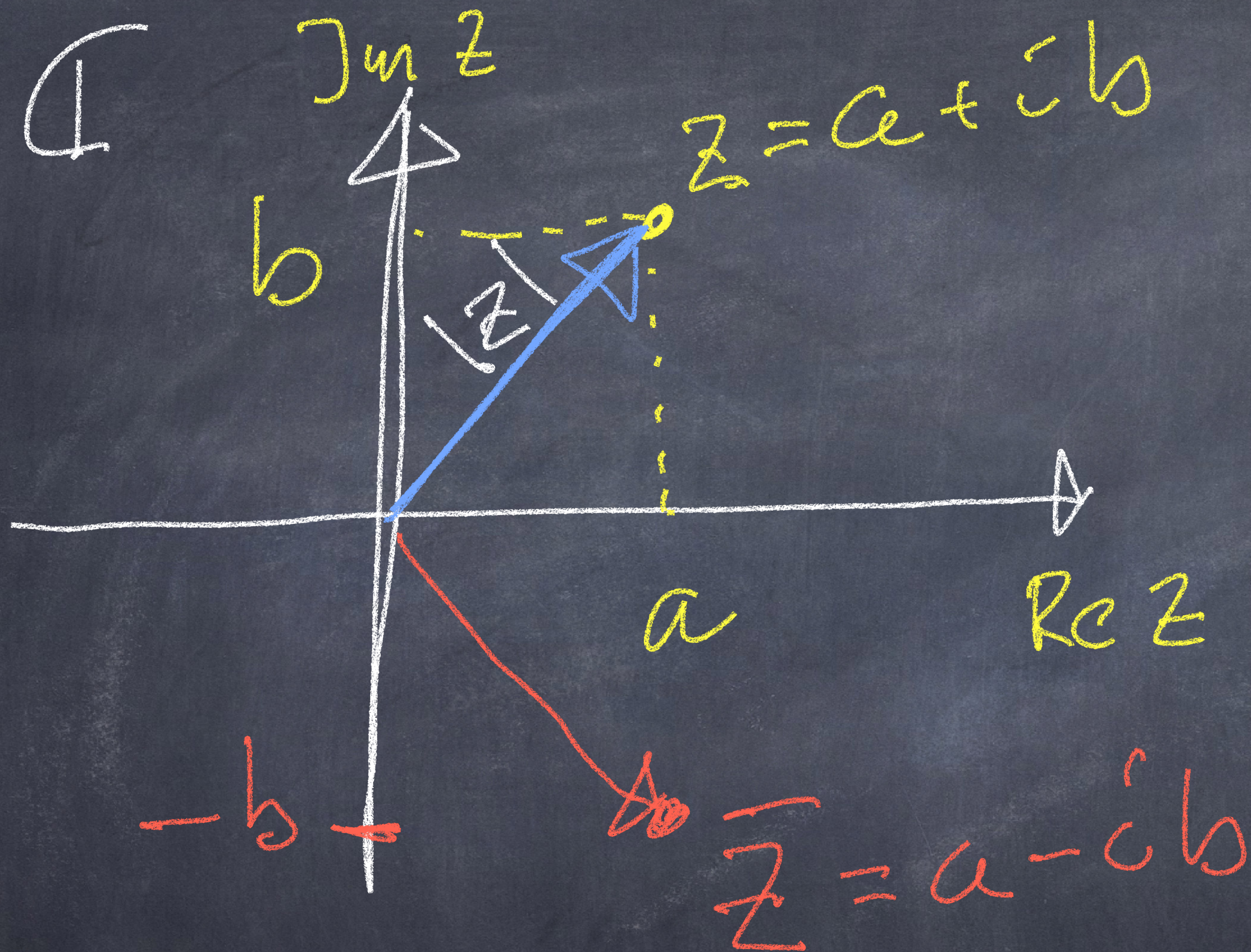
Beräkna

$$Z + W = \frac{9}{2} - i$$

$$2Z - W = 6 + 4i$$

$$Z \cdot W = \frac{11}{2} - 6i$$

$$\frac{Z}{W} = \frac{3}{10} + \frac{8}{5}i$$

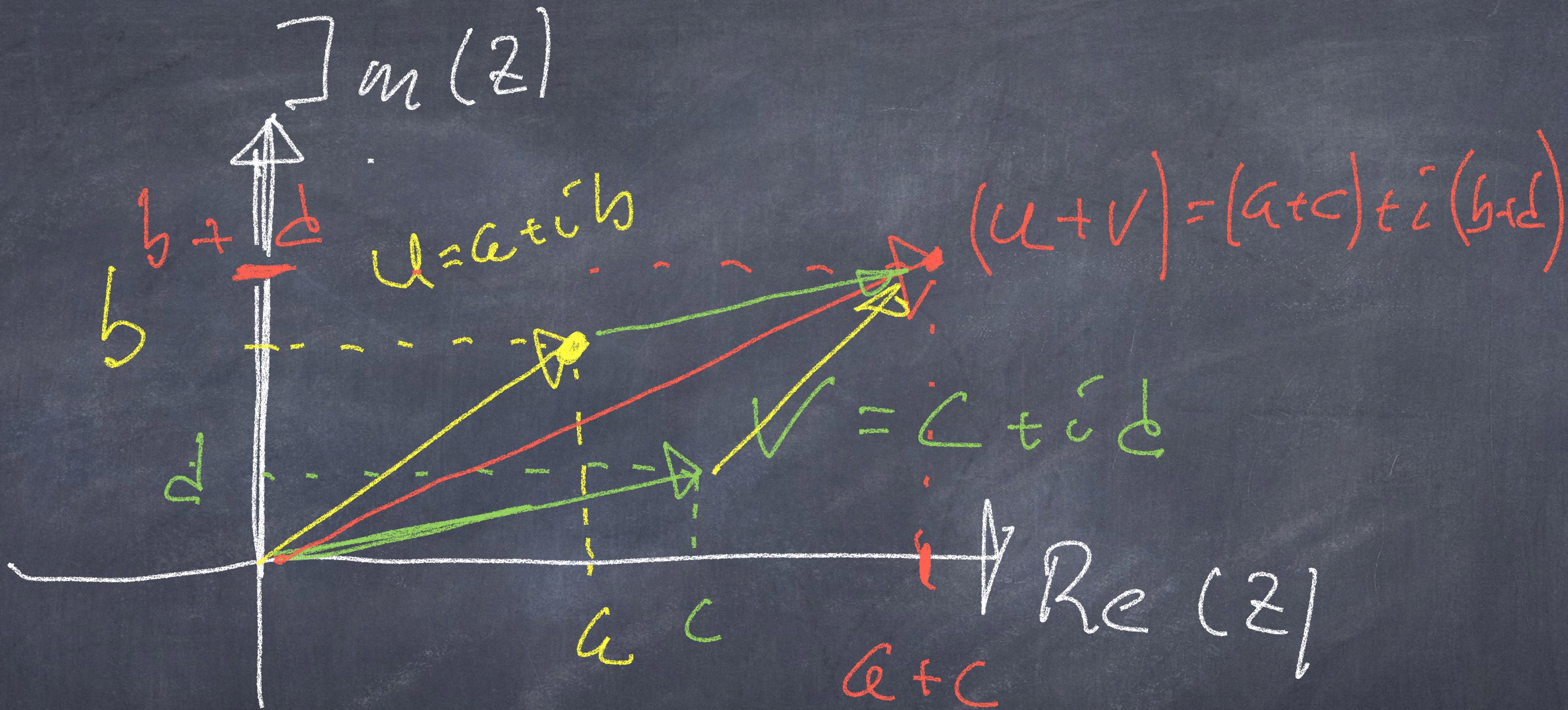


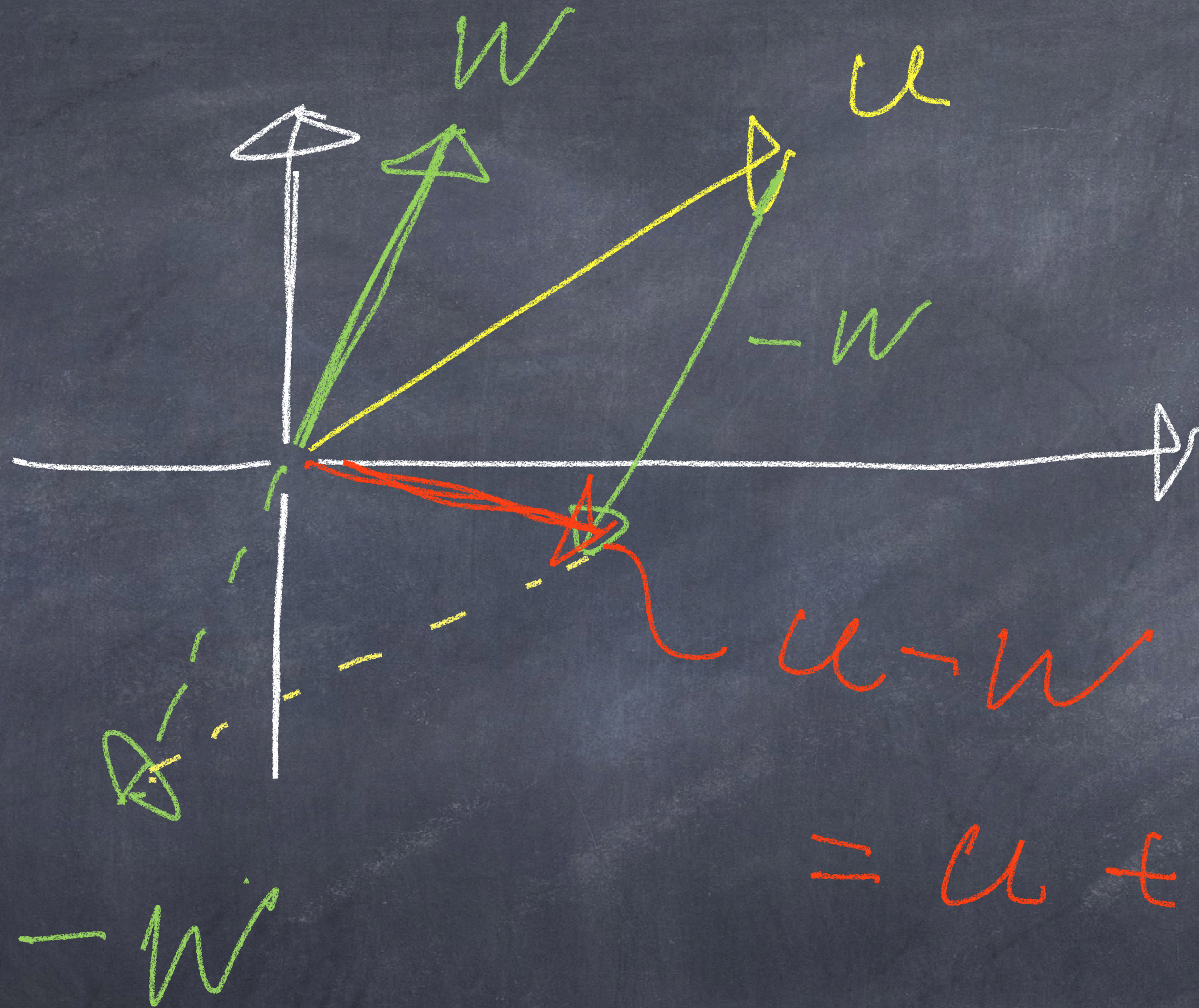
$$\text{Re}(z) = a$$

$$\text{Im}(z) = b$$

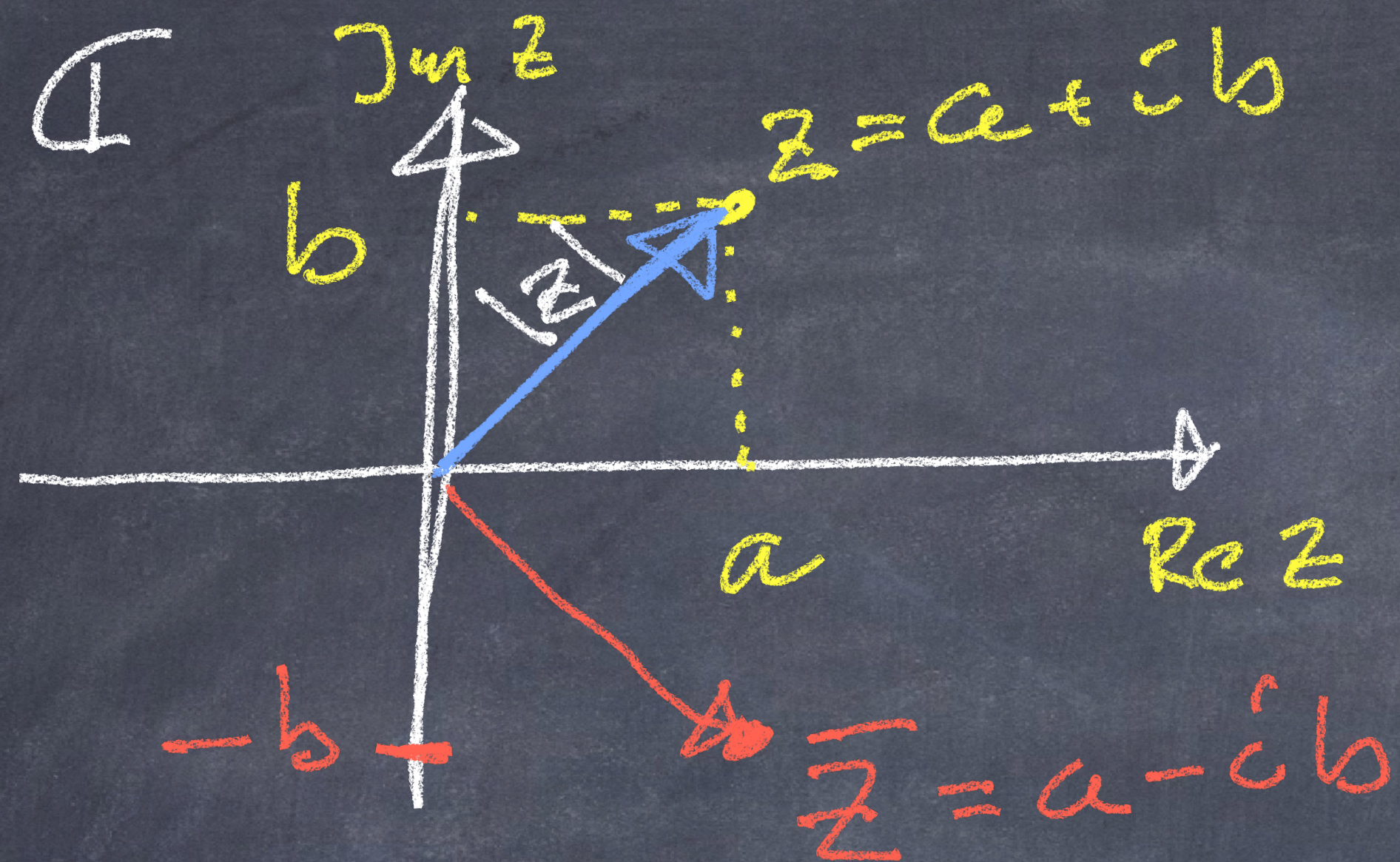
$$|z| = \sqrt{a^2 + b^2}$$

$$\neq 0$$





$$= u + (-w)$$



$$\operatorname{Re}(z) = a$$

$$\operatorname{Im}(z) = b$$

$$|z| = \sqrt{a^2 + b^2}$$

Sats:

Sats:

$$\overline{z \pm w} = \overline{z} \pm \overline{w}$$

$$\overline{z \cdot w} = \overline{z} \cdot \overline{w}$$

$$\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}} \quad (w \neq 0)$$

$$|z||w| = |zw|$$

$$\frac{|z|}{|w|} = \left| \frac{z}{w} \right|, w \neq 0$$

$$|z+w| \leq |z| + |w|$$

(trianckelolikheter)

Beweis $\left(\overline{zw} = \bar{z} \cdot \bar{w} \right)$

Läßt $z = a + ib$, $w = c + id$

$$\begin{aligned} \text{v.l.} = \overline{zw} &= \overline{(a + ib)(c + id)} = \\ &= \overline{(ac - bd) + i(ad + bc)} = (ac - bd) - i(ad + bc) \end{aligned}$$

$$\begin{aligned} \text{H.l.} &= (a - ib)(c - id) = \\ &= ac - bd + i(-ad - bc) = (ac - bd) - i(ad + bc) \\ &= \text{v.l.} \end{aligned}$$

Beris $(|zw| = |z||w|)$

V.L. & H.L. $\in \mathbb{R}$, V.L., H.L. ≥ 0 ,

Så ekvivalent att visa $|zw|^2 = |z|^2 |w|^2$.

Sätt $z = a + ib$, $w = c + id$.

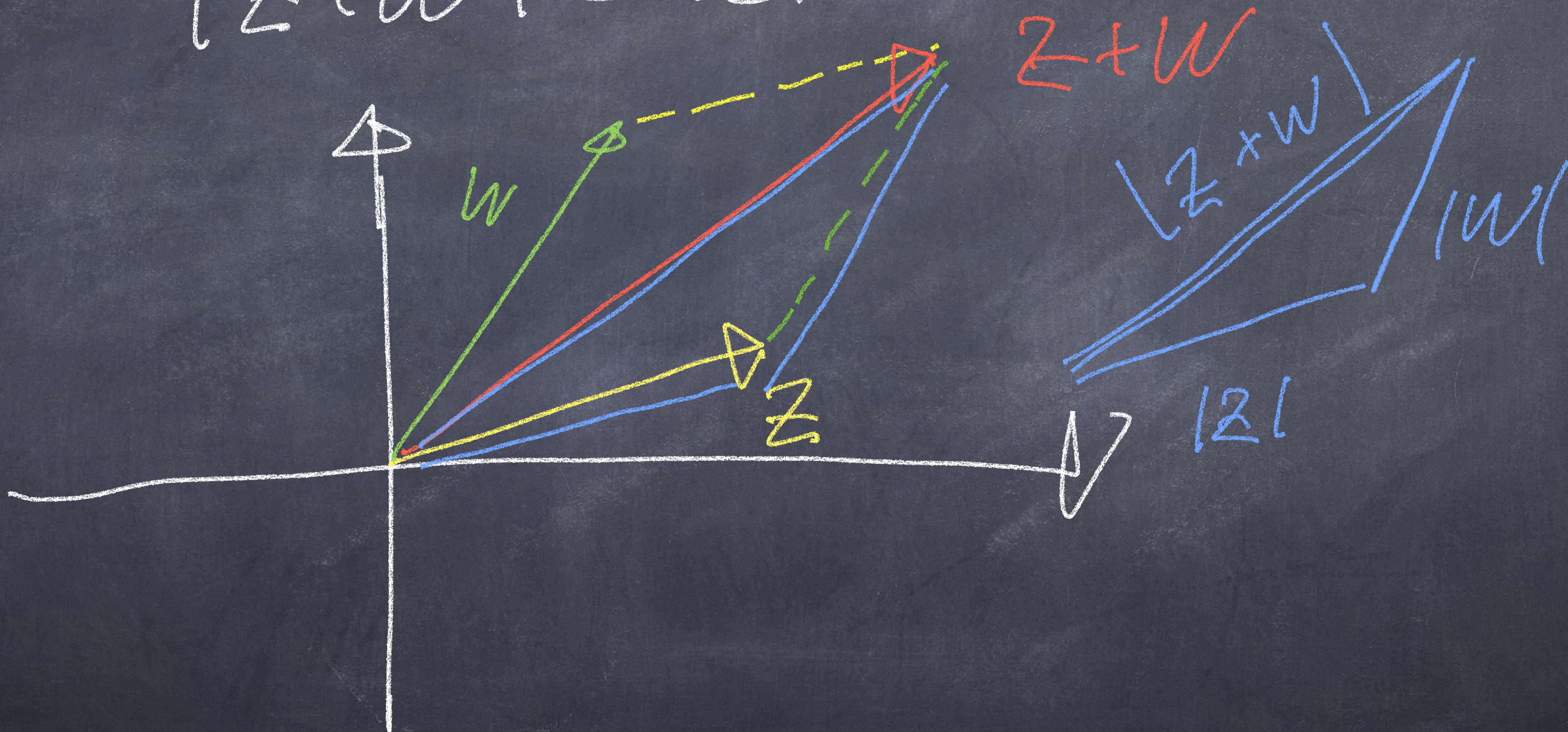
$$(V.L.)^2 = |z \cdot w|^2 = |(ac - bd) + i(ad + bc)|^2$$

$$= (ac - bd)^2 + (ad + bc)^2 = \dots$$

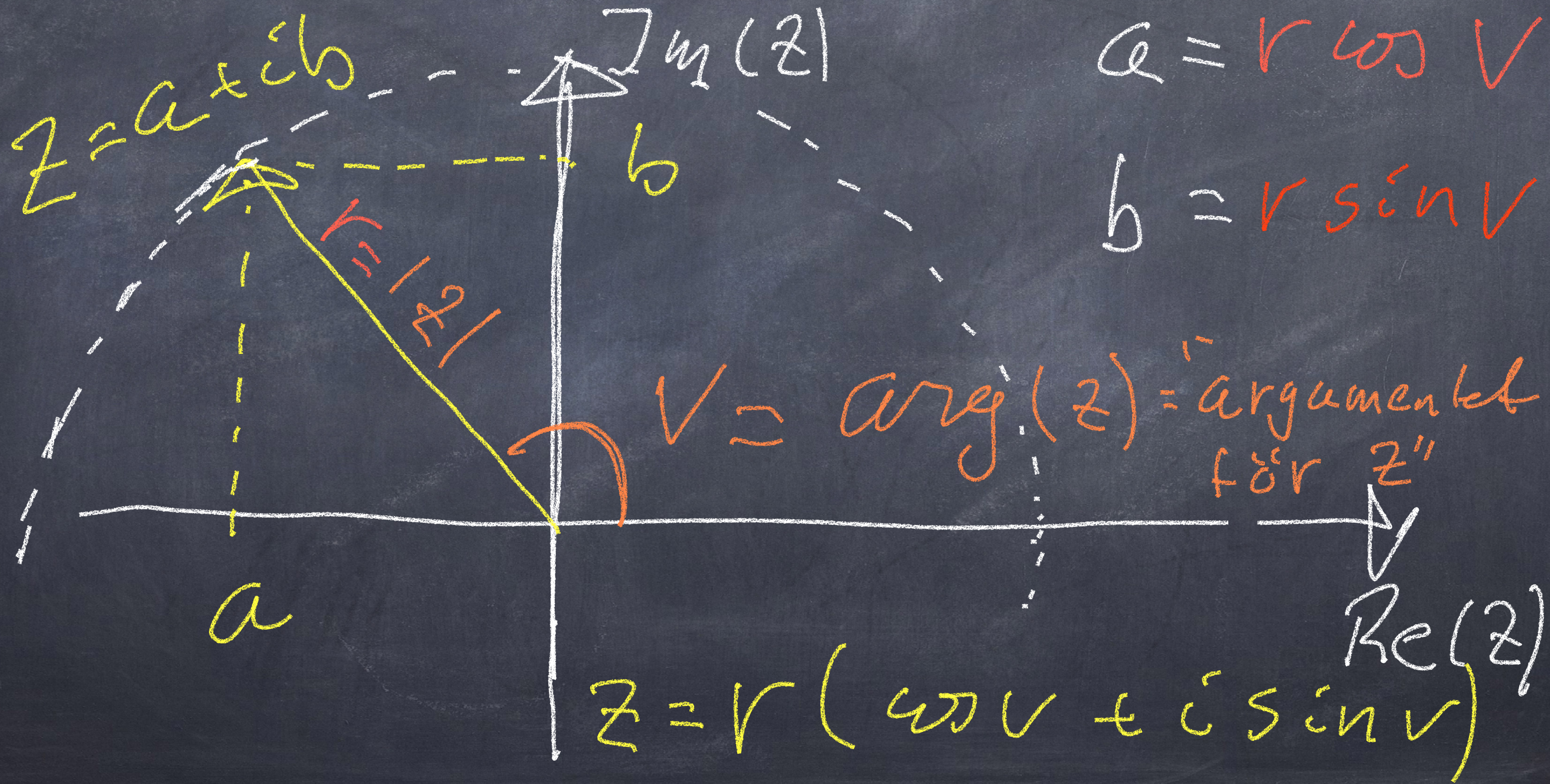
$$(H.L.)^2 = |z|^2 |w|^2 = (a^2 + b^2)(c^2 + d^2) = \dots = (V.L.)^2$$

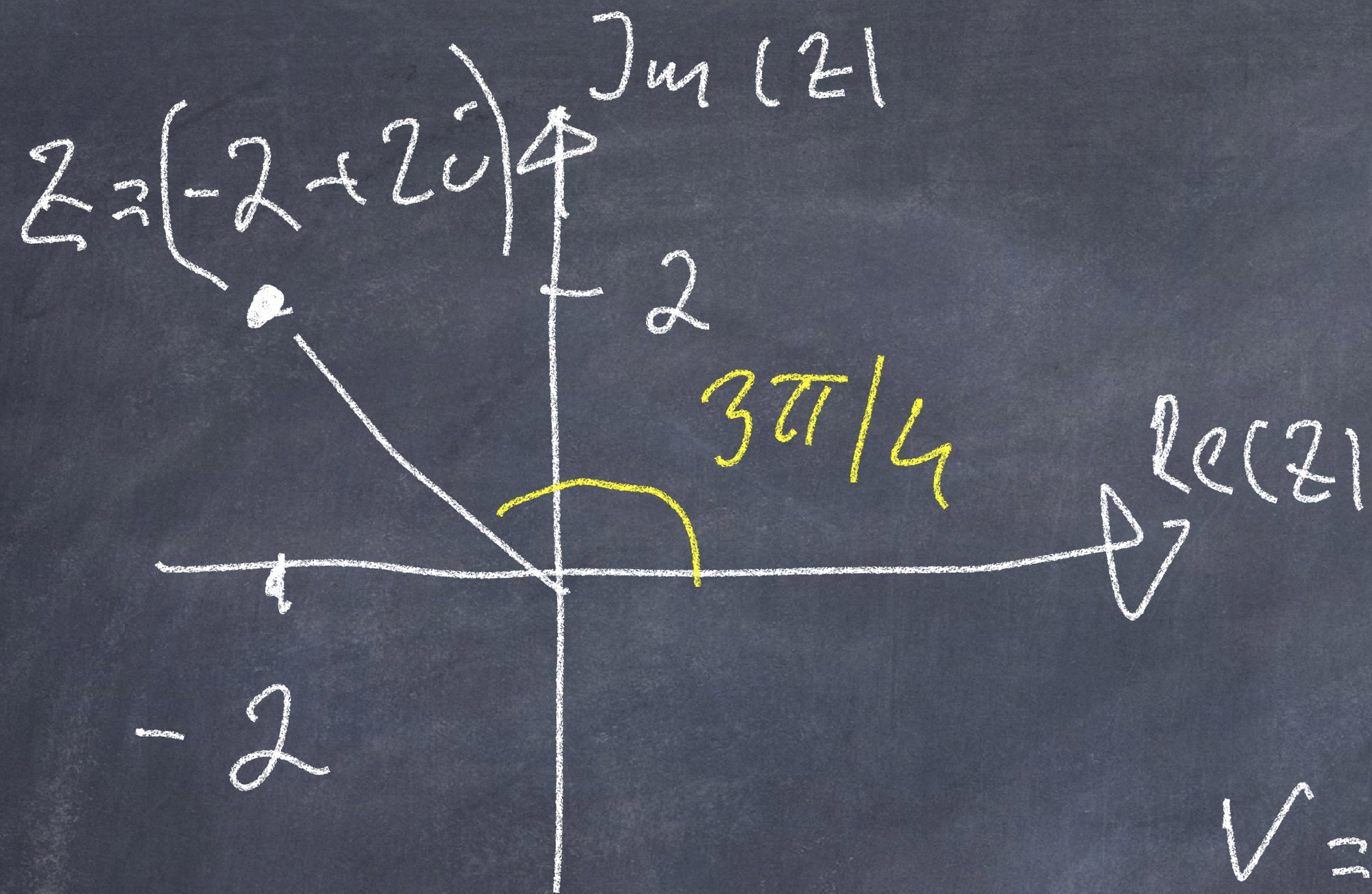
Trüngerelicheten

$$|z+w| \leq |z| + |w|$$



Polär form





$$r = |z| = \sqrt{(-2)^2 + 2^2}$$

$$= \sqrt{8} = 2\sqrt{2}$$

$$\theta = \arg(z) = \frac{3\pi}{4}$$

$$z = 2\sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$$

Skriv $z = \sqrt{3} + i$ på polär form

$$|z| = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$$

$$z = \sqrt{3} + i = 2 \left(\frac{\sqrt{3}}{2} + \frac{1}{2} i \right) =$$

Sök v s.a.

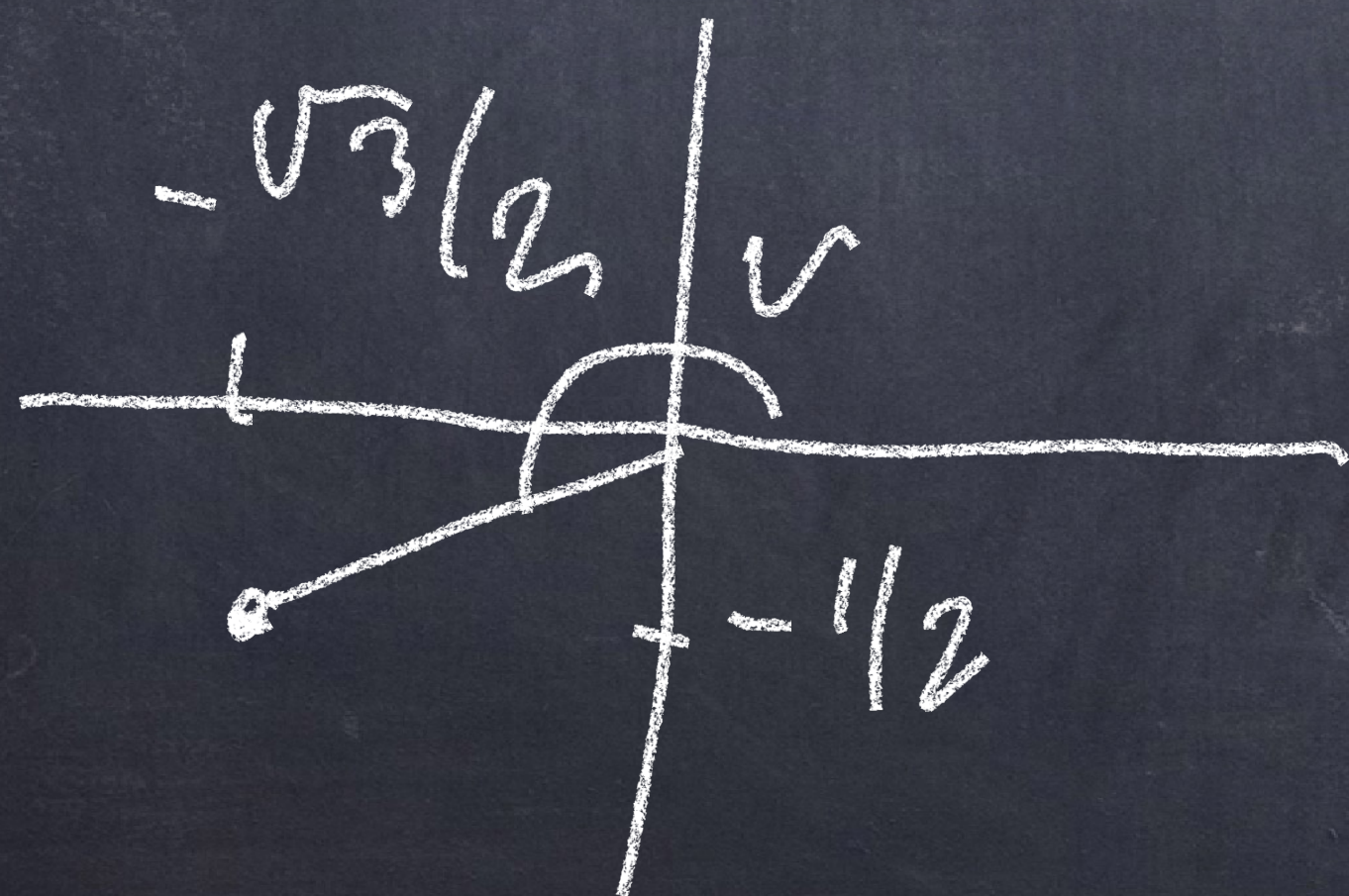
$$\begin{cases} \cos v = \sqrt{3}/2 \\ \sin v = 1/2 \end{cases} \quad \text{Ea t.c. } v = \frac{\pi}{6}$$

$$2 \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)$$

Skriv $z = -\sqrt{3} - i$ på polar form

$$|z| = \sqrt{(\sqrt{3})^2 + 1^2} = 2$$

$$z = -\sqrt{3} - i = 2 \left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i \right) = 2 \left(\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6} \right)$$



$$\varphi = \frac{7\pi}{6}$$

Polär form & multiplikation & division

$$\text{Om } z = |z| (\cos u + i \sin u)$$

$$w = |w| (\cos v + i \sin v) \quad \text{Så}$$

$$z \cdot w = |z| |w| (\cos u + i \sin u) \cdot (\cos v + i \sin v)$$

$$= |z| |w| \left(\begin{aligned} & \cos u \cos v - \sin u \sin v \\ & + i (\sin v \cos u + \cos v \sin u) \end{aligned} \right)$$

Add.

$$= |z| |w| (\cos(u+v) + i \sin(u+v))$$

formelerna

Sats: $Z = |Z|(\cos u + i \sin u)$ och
 $W = |W|(\cos v + i \sin v)$ så

• $ZW = |Z| \cdot |W| = (\cos(u+v) + i \sin(u+v))$

• $\frac{Z}{W} = \frac{|Z|}{|W|} (\cos(u-v) + i \sin(u-v))$

Exempel: $z = (1 + i) = \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$

$$w = -2 + 2i = 2\sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$$

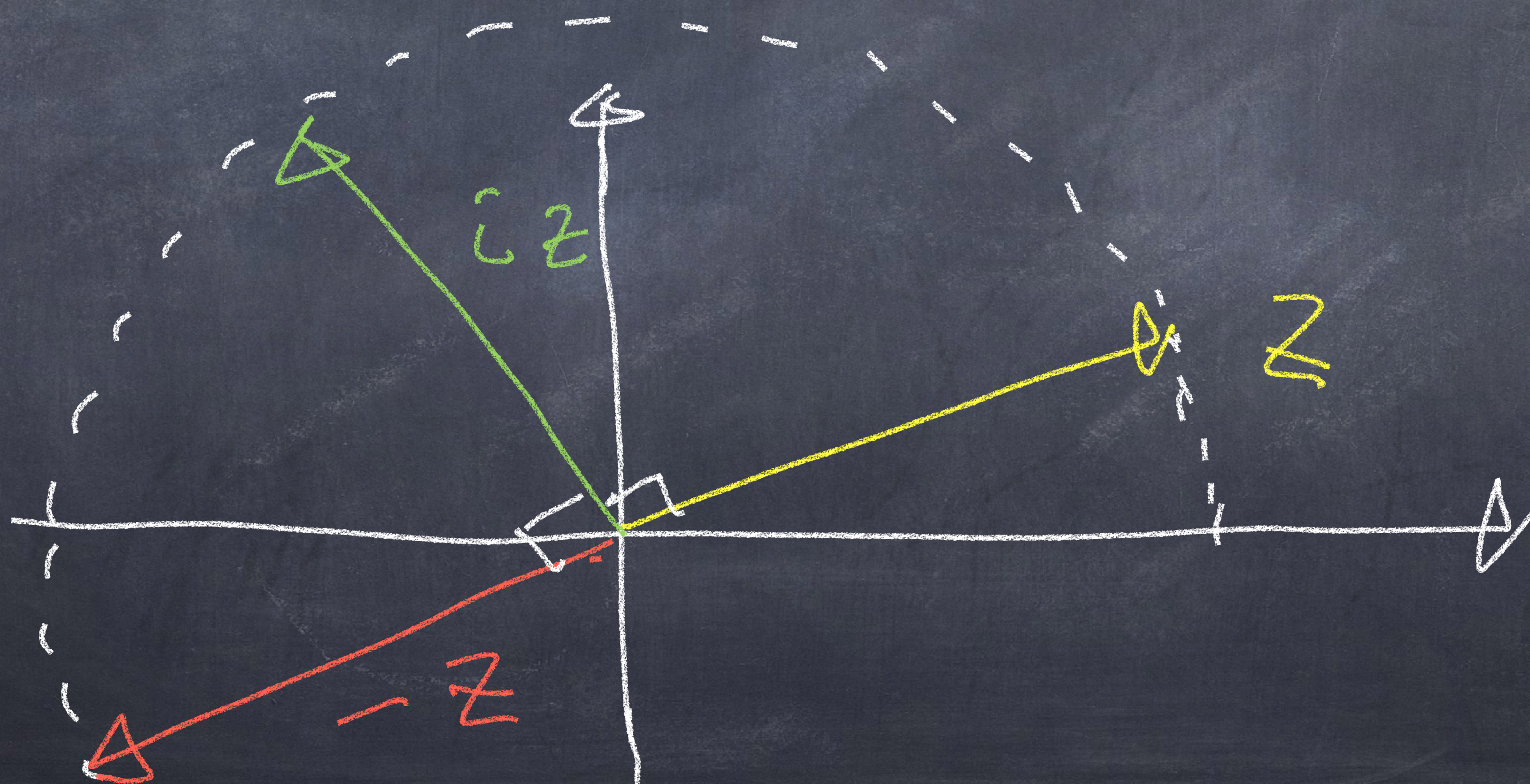
$$z \cdot w = (1 + i)(-2 + 2i) = (-2 + i \cdot 2i) + i(-2 + 2) \\ = -4$$

Polart

$$z \cdot w = \sqrt{2} \cdot 2\sqrt{2} \left(\cos \left(\frac{\pi}{4} + \frac{3\pi}{4} \right) + i \sin \left(\frac{\pi}{4} + \frac{3\pi}{4} \right) \right) \\ = 4 \left(\cos \pi + i \sin \pi \right) = -4$$

Ex: $\dot{L} = \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)$

$\dot{L}^2 = -1 = \cos \pi + i \sin \pi$



de Moïvres formel :

$$\text{Om } z = a + ib = |z|(\cos v + i \sin v)$$

$$\text{Så } z^n = (a + ib)^n =$$

$$= |z|^n (\cos nv + i \sin nv)$$

för alla $n \in \mathbb{Z}$.

Berís se su. Wikipédia
 según de Moivre's formula.

Exemple:

$$\frac{(1+i)^{10}}{(1-i)^8} = (1+i)^{10} \cdot (1-i)^{-8}$$

$$= \left[\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right]^{10} \cdot \left[\sqrt{2} \left(\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right) \right]^{-8}$$

$$= \sqrt{2}^2 \left(\cos 18\pi/4 + i \sin 18\pi/4 \right) =$$

$$= \sqrt{2}^2 (\cos 180^\circ/4 + i \sin 180^\circ/4) =$$

$$= 2 \left(\cos \left(4\pi + \frac{\pi}{2} \right) + i \sin \left(4\pi + \frac{\pi}{2} \right) \right)$$

$$= 2 \left(\underbrace{\cos \frac{\pi}{2}}_{=0} + i \underbrace{\sin \frac{\pi}{2}}_{=1} \right) = 2i$$

$$\angle a \in \quad \underline{E}(v) = \omega v + \bar{c} s \bar{c} n v$$

Erweitern

$$\underline{E}(a) \cdot \underline{E}(v) = \underline{E}(a + v)$$

$$\frac{\underline{E}(a)}{\underline{E}(v)} = \underline{E}(a - v)$$

$$[\underline{E}(a)]^n = \underline{E}(n \cdot a)$$

Vi definierar (Gl.a.) därför

$$e^{iy} = \cos y + i \sin y, \quad y \in \mathbb{R}$$

Ex.: $e^{i\pi} = \cos \pi + i \sin \pi = -1$

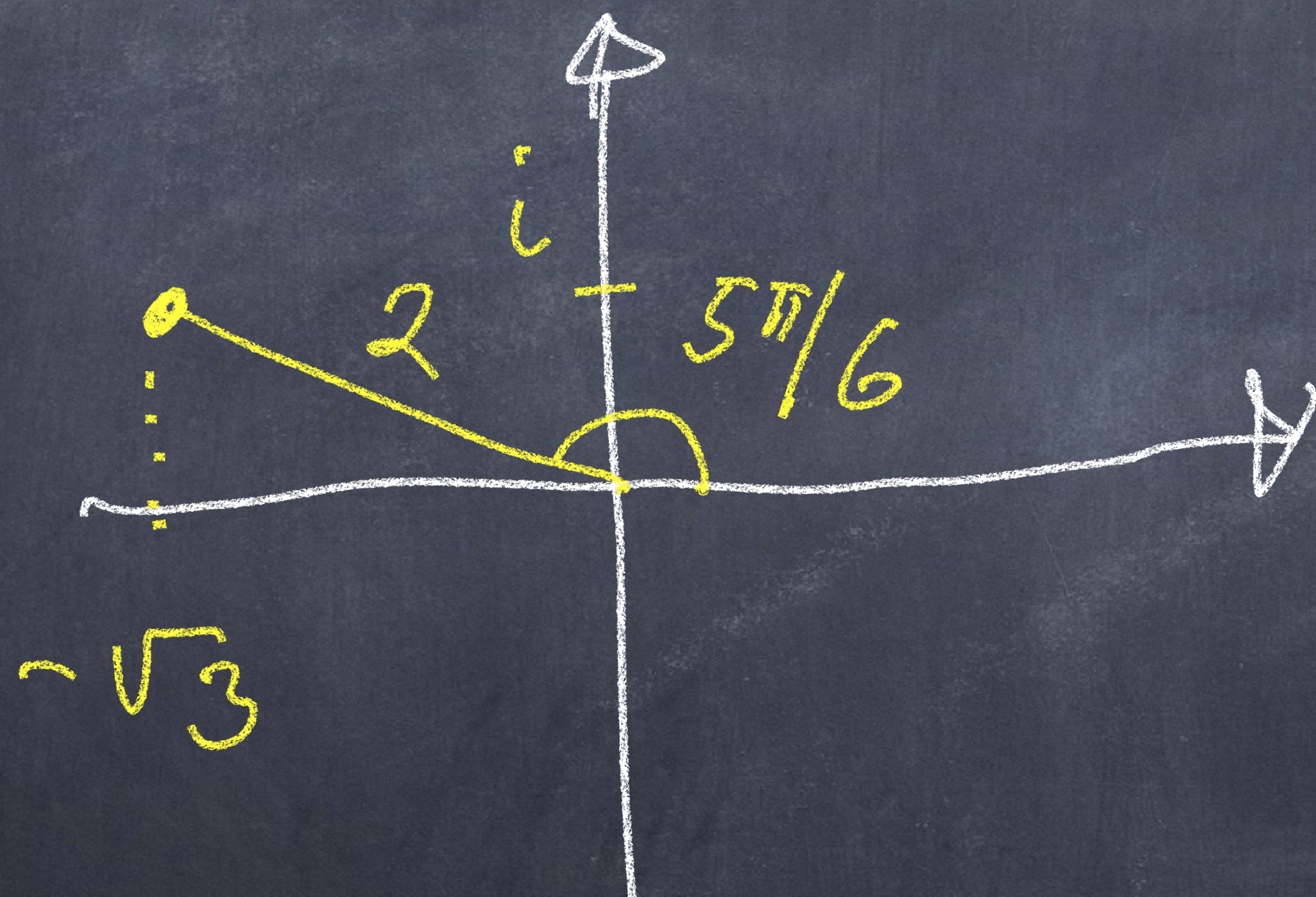
De Moivre's: $(e^{iv})^n = e^{in v}$

Ex: $-\sqrt{3} + i = 2 \left(-\frac{\sqrt{3}}{2} + \frac{1}{2} i \right) = 2 \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)$

$\Rightarrow 2 e^{i 5\pi/6}$

$= e^{\ln 2} \cdot e^{i 5\pi/6}$

$= e^{\ln 2 + i 5\pi/6}$



$$Om \quad z = x + iy$$

$$e^z = e^{x+iy} = e^x \cdot e^{iy}$$

$$= e^x (\cos y + i \sin y)$$

$$dvs \quad |e^{x+iy}| = e^x, \quad \arg(e^{x+iy}) = y$$

$$\text{Ex: } Om \quad z = \ln 2 + i \frac{5\pi}{6}, \quad \text{så är}$$

$$\begin{aligned} e^z &= e^{\ln 2 + i 5\pi/6} = e^{\ln 2} e^{i 5\pi/6} = 2 \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right) \\ &= -\sqrt{3} + i \end{aligned}$$

Komplexa trigonometrisk

$$\begin{cases} e^{iy} = \cos y + i \sin y \\ e^{-iy} = \cos y - i \sin y \end{cases} \Rightarrow \begin{cases} \cos y = \frac{e^{iy} + e^{-iy}}{2} \\ \sin y = \frac{e^{iy} - e^{-iy}}{2i} \end{cases}$$

för $y \in \mathbb{R}$

$$\begin{cases} \cos z \stackrel{\text{def}}{=} \frac{e^{iz} + e^{-iz}}{2} \\ \sinh z \stackrel{\text{def}}{=} \frac{e^{iz} - e^{-iz}}{2i} \end{cases}$$

Fördjupning / ^u Överkurs



$$\cos(x+iy) =$$

$$\frac{1}{2} \left(e^{i(x+iy)} + e^{-i(x+iy)} \right)$$

$$= \frac{1}{2} \left(e^{-y+ix} + e^{y-ix} \right)$$

$$= \frac{1}{2} e^{-y} (\cos x + i \sin x) + \frac{1}{2} e^y (\cos x - i \sin x)$$

$$\cosh z = \cosh(\bar{z})$$

$$i \sinh z = \sinh(\bar{z})$$

$$\cosh iz = \sinh z$$

$$i \sinh iz = -\sinh z$$

$$\cosh t = \frac{e^t + e^{-t}}{2}$$

läßt $z \in \mathbb{C}$,

$$\sinh t = \frac{e^t - e^{-t}}{2}$$

Kann man zeigen $(x \in \mathbb{R})$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$x = iy$ setzen $e^{iy} = \cos y + i \sin y$