Lecture 1

1. Real numbers and the real line

Set of natural numbers:

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

If $a, b \in \mathbb{N}$ then $a+b \in \mathbb{N}$
and $a \cdot b \in \mathbb{N}$

This is not true for subtraction and division

Set of integers:

$$\mathbb{Z} = \{\dots -3, -2, -1, 0, 1, 2, 3, \dots\}$$

Note that $a-b \in \mathbb{Z}$ for all $a, b \in \mathbb{Z}$

The rational numbers:

$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \neq 0 \right\}$$

↳ ratio between two integers

Example: $\frac{1}{2}, -\frac{2}{3}, -\frac{9}{1}$

<u>Remark:</u> $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$

All the integers are included in the rational numbers since any number a can be written as the ratio $\frac{a}{1}$.

The irrational numbers : the number that can not be written as a ratio

<u>Example</u> $x^2 = 2$

What number times itself equals 2?

The solution is $x = \pm\sqrt{2}$

$\sqrt{2} = 1,4142....$ no repeating pattern

$\sqrt{2} \neq \frac{m}{n}$ for all m, n integers.

Other famous irrational numbers:

- The golden ratio: $\frac{1+\sqrt{5}}{2} = 1,618...$

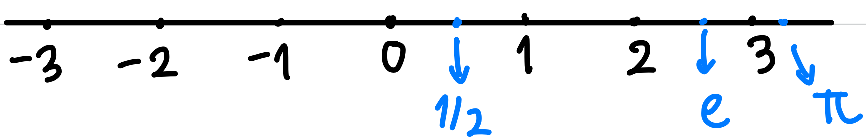
- The ratio of the circumference of a circle to its diameter: $\pi = 3{,}14159\ldots$
- The most important number in calculus $e = 2{,}7182\ldots$

Irrational numbers can be further subdivided into

- algebraic numbers $\left(\sqrt{2}, \dfrac{1+\sqrt{5}}{2}, \ldots\right)$
- transcendental numbers $(\pi, e, \ldots)$

Real numbers $\mathbb{R}$ = all rational numbers and all irrational numbers.

The real numbers are "all the numbers" on the real line (uncontable infinity)

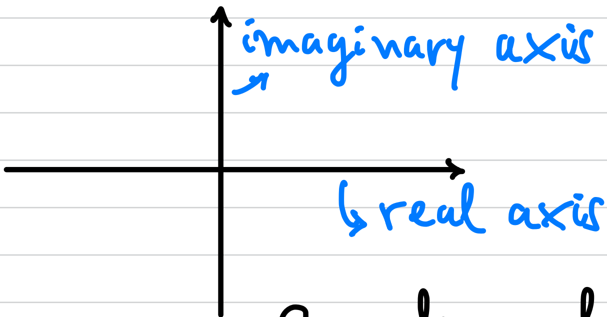


Complex numbers $\mathbb{C} = \{ a + bi ,\ a, b \in \mathbb{R} \}$

i is the imaginary unit $\sqrt{-1}$

If $a \in \mathbb{R}$ then $a = a + 0.i$

thus $\mathbb{R} \subset \mathbb{C}$.



Complex plane

2. <u>Properties of the real numbers</u>:

Algebraic properties

a. Closure : $a+b$, $a.b$ are real.

b. Associative

$$(a+b) + c = a + (b+c)$$
$$(a.b).c = a.(b.c)$$

c. Commutative

$$a + b = b + a$$
$$a.b = b.a$$

d. Distributive

$$a.(b+c) = a.b + a.c$$

e. Identity

$$a + 0 = 0 + a = a$$
$$1.a = a.1 = a$$

f. Inverse

$$a + (-a) = 0$$
$$a.\frac{1}{a} = 1 \text{ if } a \neq 0$$

g. Multiplicative property of zero

$$a.0 = 0.a = 0$$

h. Negation

$$-(-a) = a$$
$$(-a)(-b) = a.b$$

The order properties: for any a, b either $a > b$, $a < b$ or $a = b$.

Rules for inequalities

- If $a>b$ and $b>c$ then $a>c$.
- If $a>b$ then $a+c>b+c$ for all c.
- If $a>b$ and $c>0$ then $a.c > b.c$
- If $a>b$ and $c<0$ then $a.c < b.c$

Ex: Find all x such that

$$\frac{2x}{1-x} > 1$$

Note that $1-x \neq 0$

If $1-x>0$ then $2x > 1-x$

$\Rightarrow 1>x$ and $x>\frac{1}{3} \Rightarrow x \in \left(\frac{1}{3}, 1\right)$

If $1-x<0$ then $2x < 1-x$

$\Rightarrow 1<x$ and $x<\frac{1}{3} \Rightarrow$ no solution

- If $a>b>0$ then $\frac{1}{a} < \frac{1}{b}$

3. Intervals = is a subset of the real line

infinite interval $(-\infty, a)$ open interval (a, b)

- Open interval from a to b

$$(a, b) = \{ x \in \mathbb{R} : a < x < b \}$$

- Closed interval

$$[a, b] = \{ x \in \mathbb{R} : a \leq x \leq b \}$$

- Half-open interval $[a, b)$ or $(a, b]$
- Infinite interval

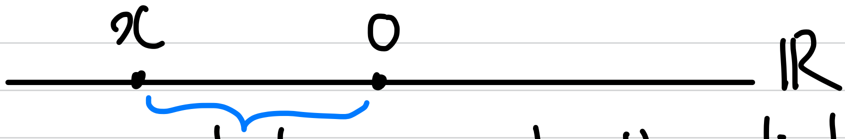
$$(a, +\infty) = \{ x \in \mathbb{R} : x > a \}$$

$$(-\infty, a) = \{ x \in \mathbb{R} : x < a \}$$

$$(-\infty, +\infty) = \mathbb{R}$$

4. <u>Absolute value</u> (magnitud) is defined by the formula

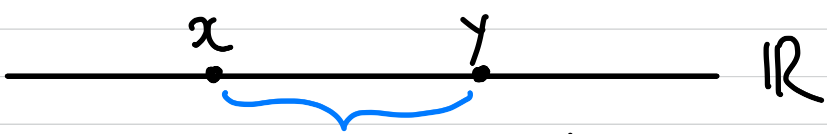
$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$



$|x|$ represents the distance from x to 0 in the real line.

<u>Ex:</u> $|-2| = 2$, $|1| = 1$

$|0| = 0$, $|-\pi| = \pi$



$|x-y|$ distance between two points x and y.

Properties of absolute value:

- $|-a| = |a|$

- $|ab| = |a| \cdot |b|$ and $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, $b \neq 0$

- $|a \pm b| \leq |a| + |b|$

(the triangle inequality)

See the textbook for proofs

<u>Example</u> Find all x such that

$$|x+2| > 5 - 2x$$

We have two cases:

Case 1: $x + 2 > 5 - 2x$ if $x + 2 \geq 0$

$\Rightarrow$ $x > 1$ and $x \geq -2$

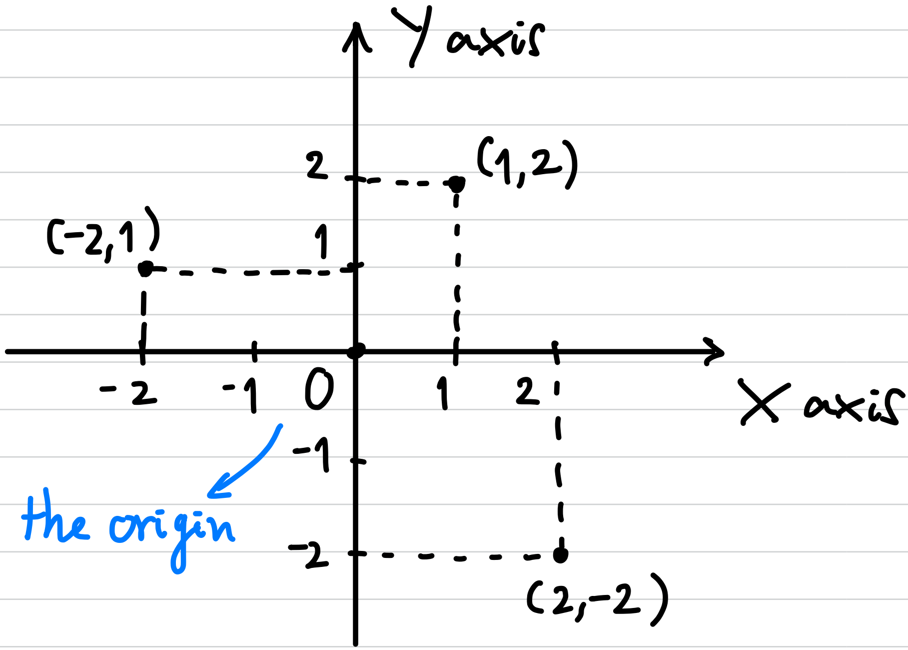
$x \in (1, +\infty)$ is the solution

Case 2: $-(x+2) > 5 - 2x$ if $x + 2 < 0$

$\Rightarrow$ $x > 7$ and $x < -2$

no solution!

5. Cartesian Coordinates in the plane



Increments and distances: consider a particle that moves from $A(x_1, y_1)$ to $B(x_2, y_2)$.

The increments are

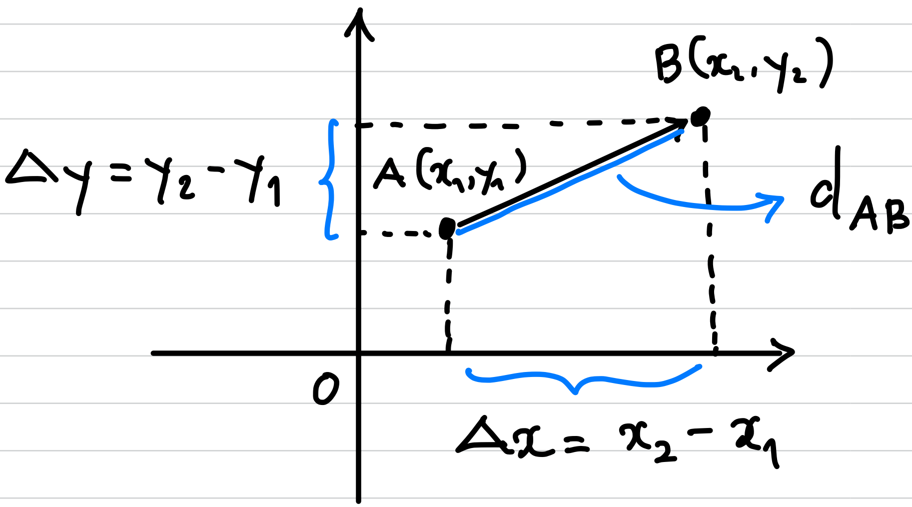
$$\Delta x = x_2 - x_1 \quad \text{and}$$

$$\Delta y = y_2 - y_1$$

The distance between A and B is

$$d_{AB} = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

(Pythagorean Theorem)



<u>Example:</u> $A(-1, 1)$ and $B(-2, 3)$

The increments from A to B are

$$\Delta x = -2 - (-1) = -1$$
$$\Delta y = 3 - 1 = 2$$

The distance $d_{AB} = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$.