SF 1684 Algebra and Geometry
Lecture 20

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Topics for Today

1. Abstract Vector Space
2. Linear Transformations of Abstract Vector Spaces
3. Isomorphisms of Abstract Vector Spaces

Axioms of a Vector Space

Recall from Lecture 1, that we defined a vector space as something that satisfies these axioms

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9. (Distributivity) For every $c, d \in F$ and every $\vec{u}, \vec{v} \in V$, $(c + d) \cdot \vec{u} = c \cdot \vec{u} + d \cdot \vec{v}$ and $c \cdot (\vec{u} + \vec{v}) = c \cdot \vec{u} + c \cdot \vec{v}$

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Theorem

If $\vec{v}$ is a vectors in a vector space V, and if k is a scalar, then

1. $0\vec{v} = \vec{0}$
2. $k\vec{0} = \vec{0}$
3. $(-1)\vec{v} = -\vec{v}$

Vector Space of Functions

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Now, we can begin to talk about the properties of vectors spaces we have dealt with. That is: linear dependence, subspaces, basis, linear transformations, etc...

Exercise

Let 1 denote the constant function that sends everything to 1. Show that the set $\{\underbrace{1}_{\vec{v}_1}, \underbrace{\cos^2(x)}_{\vec{v}_2}, \underbrace{\sin^2(x)}_{\vec{v}_3}\}$ is a linear **dependent** set of vectors in the vectors space of functions.

Three vectors are linearly dependent iff there exists $c_1, c_2, c_3 \neq 0$ such that $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = 0$

Can I find c_1, c_2, c_3 such that $* \; c_1 \cdot 1 + c_2\cos^2 x + c_3\sin^2 x = \underset{\text{the zero function}}{0}$

i.e. $*$ must be true for all x.

$c_1 = -1$, $c_2 = 1$, $c_3 = 1$

$$c_1 \cdot 1 + c_2\cos^2 x + c_3\sin^2 x = -1 + \underbrace{\cos^2 x + \sin^2 x}_{\text{for all } x} = -1 + 1 = 0$$

Linear Dependence

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Wronski's Test

If we have a set of functions from $\mathbb{R} \to \mathbb{R}$ given by

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then we define the **Wronskian of the functions** to be

$$\underset{\text{2nd derivative}}{W(x)} := \det \left(\underbrace{\begin{array}{l} \begin{pmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ f_1'(x) & f_2'(x) & \dots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{pmatrix} \end{array}}_{M} \right) \begin{array}{l} \text{functions} \\ \text{derivative of functions} \\ \\ \text{(n-1)-st derivative function} \end{array}$$

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Theorem (Wronski's Test)

A set of n functions from $\mathbb{R} \to \mathbb{R}$ are linearly independent if and only if the Wronskian of the functions is not identically zero.

Example of Wronski's Test

Exercise

Using that fact that if $f_1(x) = 1$, $f_2(x) = \cos^2(x)$ and $f_3(x) = \sin^2(x)$, then

$$f_1' = 0, f_1'' = 0, f_2' = -2\sin(x)\cos(x), f_2'' = 2\sin^2(x) - 2\cos^2(x)$$

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show that $\{f_1, f_2, f_3\}$ is linearly **dependent** by showing that the Wronskian is identically zero.

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Setting up the Wronskian, we see that

$$W(x) = \det\left(\begin{array}{cl} \begin{pmatrix} 1 & \cos(x) & \sin(x) \\ 0 & -2\sin(x)\cos(x) & 2\sin(x)\cos(x) \\ 0 & 2\sin^2(x) - 2\cos^2(x) & 2\cos^2(x) - 2\sin^2(x) \end{pmatrix} & \begin{array}{l} \text{functions} \\ \text{derivatives} \\ \text{2nd deriv} \end{array} \end{array}\right)$$

Example of Wronski's Test 2

Expanding the determinant along the first column, we find that

$$W(x) = \det\left(\begin{pmatrix} -2\sin(x)\cos(x) & 2\sin(x)\cos(x) \\ 2\sin^2(x) - 2\cos^2(x) & 2\cos^2(x) - 2\sin^2(x) \end{pmatrix}\right)$$

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$$= 0$$

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Example: If we let W_{n-1} be the set of all polynomials of degree at most $n-1$:

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Linear Independent Polynomials

Exercise

Using the fact that if $f_j(x) = x^j$ then $f_m^{(m)}(x) = m!$ and $f_j^{(m)}(x) = 0$ if $\cancel{j \geq m}$, show that the set $\{f_0, f_1, \dots, f_{n-1}\}$ is linear independent for any n.

$m > j$

$\{1, x, x^2, \dots, x^{n-1}\}$

if there is a $c_0, c_1, \dots, c_{n-1}$ s.t. $c_0 \cdot 1 + c_1 \cdot x + \dots + c_{n-1} x^{n-1} = 0$

as function

for all x $\implies$ $c_0, c_1, \dots, c_{n-1} = 0$

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Using the fact, we see that the Wronskian of the vectors will be

$$W(x) = \det \begin{array}{c} \begin{array}{cccccc} f_0 & f_1 & f_2 & f_3 & & f_{n-1} \end{array} \\ \left(\begin{pmatrix} 1 & x & x^2 & x^3 & \cdots & x^{n-1} \\ 0 & 1 & * & * & \cdots & * \\ 0 & 0 & 2 & * & \cdots & * \\ 0 & 0 & 0 & 6 & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (n-1)! \end{pmatrix}\right) \end{array} \begin{array}{l} \text{function} \\ \text{derivative} \\ \\ \\ \\ \\ \end{array}$$

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$$W(x) = \det\left(\begin{pmatrix} 1 & x & x^2 & x^3 & \cdots & x^{n-1} \\ 0 & 1 \; f_1^{(1)} & * & * & \cdots & * \\ 0 & 0 & 2 \; f_2^{(2)} & * & \cdots & * \\ 0 & 0 & 0 & 6 \; f_3^{(3)} & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (n-1)! \end{pmatrix}\right)$$

$$= 1 \times 1 \times 2 \times 6 \times \cdots \times (n-1)!$$

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$$= 1 \times 1 \times 2 \times 6 \times \cdots \times (n-1)! \neq 0$$

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Moreover, clearly any polynomials of degree at most $n-1$ can be written as a linear combination of vectors in $\{1, x, x^2, \ldots, x^{n-1}\}$ and so it is a spanning set.

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Hence, if $W_{n-1} = \{a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} : a_i \in \mathbb{R}\}$, then

$$\dim(W_{n-1})$$

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Hence, if $W_{n-1} = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} : a_i \in \mathbb{R}\}$, then

$$\dim(W_{n-1}) = \text{ number of elements in a basis } = n$$

Infinite Dimensional Vector Space

However, what if we want to consider the set of polynomials of any degree $W = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n : a_i \in \mathbb{R}, n \geq 0\}$.

Infinite Dimensional Vector Space

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Then we see that a basis for this would necessarily be all the powers x: $\{1, x, x^2, x^3, \dots\}$.

$\{1, x, x^2, \dots x^n\}$ is linear independent for all n

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Moreover, since all polynomials are also functions, we see that the vector space of all functions from the reals to the reals is also infinite dimensional.

Question: What is the basis of all functions?

Unusual Vector Space

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Exercise

Let V be the set of positive real numbers but define vector addition and scalar multiplication by $\mathbb{R}$ as follows:

multiplication of real numbers

$k \in \mathbb{R}$

$$u \oplus v = u \cdot v \text{ (vector addition)}$$

$$k \otimes u = u^k \text{ (scalar multiplication by } \mathbb{R})$$

u, v are vectors in V so u & v, in particular are positive reals

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What is $\vec{0}$?
What is $(-u)$?

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Show that these operations satisfy the axioms and hence makes V a vector space.

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Multiplying as real numbers

adding as vectors in V

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$$2 \oplus 1 = 2 \cdot 1 = 2$$

adding the vectors in V

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V is the set of positive real numbers

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scalar multiplication of vectors in V

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$k(u+v) = ku + kv$

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Even though the set of element in V are the real numbers, V is **NOT** a subspace of $\mathbb{R}$ or any other $\mathbb{R}^n$! This is because in order to be a subspace, the vector addition and scalar multiplication must be the *same* in both spaces!! Clearly the vector addition and scalar multiplication in V and $\mathbb{R}$ are different.

Linear Transformations Between Abstract Vector Spaces

Definition

If $T : V \to W$ is a function from a vector space V to a vector space W then T is called a **linear transformation** from V to W if the following properties hold for all vectors $\vec{u}, \vec{v}$ and for all scalars c

1. $T(c\vec{u}) = cT(\vec{u})$
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Theorem

If $T : V \to W$ is a linear transformation, then:

1. $T(\vec{0}) = \vec{0}$
2. $T(-\vec{u}) = -T(\vec{u})$
3. $T(\vec{u} - \vec{v}) = T(\vec{u}) - T(\vec{v})$

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Theorem

If $T : V \to W$ is a linear transformation then $\ker(T)$ *is subspace of* V *and* $\operatorname{ran}(T)$ *is a subspace of* W.

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A linear transformation $T : V \to W$ is one-to-one if it maps distinct vectors in V into distinct vectors in W.

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The difference in bijection & isomorphism comes into play when considering infinite dimensional vector spaces

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in particular finite dimensional

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Let V be an n-dimensional vectors space. Then there is a basis for V: $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$.Then the linear transformation defined by

$$T(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n) = a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n \in \mathbb{R}^n$$

is an isomorphism. check: that this a linear transformation & one-to-one and onto ☐

Examples

Let $V = \{f : \mathbb{R} \to \mathbb{R}\}$ and let $x_1, x_2 \ldots, x_n$ be any set of real numbers.

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Exercise: prove this is linear

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$f(x) = (x-x_1)(x-x_2)\cdots(x-x_n)$

$T(f) = \vec{0}$

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$T(f) = \vec{0} = (0, \ldots 0)$

$(f(x_1), f(x_2), \ldots, f(x_n))$ $\Longleftrightarrow$ $f(x_1) = 0$, $f(x_2) = 0$, $\ldots$ $f(x_n) = 0$

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The kernel would be any function that is 0 at all of $x_1, \ldots, x_n$. So it is not one-to-one.

If all the x_i were distinct then the range would be all of R^n. So it would be onto.

Suppose $x_1 = x_2$ then $f(x_1) = f(x_2)$ and therefore can not be anything in $\mathbb{R}^n$ with different 1^{st} & 2^{nd} coordinate.

Examples

Let $W_{m-1} = \{a_0 + a_1x + a_2x^2 + \cdots + a_{m-1}x^{m-1}\}$ be the space of polynomials of degree at most $m - 1$

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$f(x) = 1$ $f(x_i) = 1$ $f(x_1) = 1$

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$f(x) = x^i$
$f(x_1) = x_1^i \ldots$

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Matrix of Evaluation Function

Thus we see that T can be given by the matrix

$$\begin{array}{c}
\begin{pmatrix}
1 & x_1 & x_2 & \dots & x_n \\
1 & x_1^2 & x_2^2 & \dots & x_n^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_1^{m-1} & x_2^{m-1} & \dots & x_n^{m-1}
\end{pmatrix} \\
\begin{array}{ccccc}
T(1) & T(x) & T(x^2) & & T(x^{m-1})
\end{array}
\end{array}$$

$$\begin{bmatrix}
1 & x_1 & x_1^2 & \dots & x_1^{m-1} \\
\vdots & x_2 & x_2^2 & & \vdots \\
 & \vdots & \vdots & & \\
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For any real number $c_1, c_2 \ldots, c_n, x_1, x_2, \ldots, x_n$ you can find a polynomial of degree n such that

Conclusion About Polynomials

for all $\vec{c} \in \mathbb{R}^n$ I can find $f \in W_n$ s.t

$$T(f) = \vec{c}$$

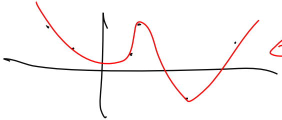
Hence we may conclude the following theorem about polynomials

Theorem

For any real number $c_1, c_2 \ldots, c_n, x_1, x_2, \ldots, x_n$ you can find a polynomial of degree n such that

$$f(x_1) = c_1, f(x_2) = c_1, \ldots, f(x_n) = c_n$$

$(x_1, c_1), (x_2, c_2) \ldots (x_n, c_n)$ as points on the plane



can find polynomial that goes through all the points

More Examples

The map from the space of functions to itself that takes the derivative is also a linear transformation:

$$T : V \to V$$
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Take a polynomial of degree n and sends to a polynomial of degree $n-1$.

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$$
\begin{pmatrix}
0 & \underline{1} & 0 & \dots & 0 \\
0 & 0 & \underline{2} & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \\
0 & 0 & 0 & \dots & \underline{n-1} \\
0 & 0 & 0 & \dots & 0
\end{pmatrix}
$$

a row of zeros

Final Example

The trace function from the $n \times n$ square matrices to $\mathbb{R}$ is also a linear transformation:

vector space

$$T : M_{n,n} \to \mathbb{R}$$
$$A \to \mathrm{Tr}(A)$$

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The trace function from the $n \times n$ square matrices to $\mathbb{R}$ is also a linear transformation:

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The determinant function from the $n \times n$ square matrices to $\mathbb{R}$ is *not* a linear transformation:

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\end{aligned}
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since $\det(cA) = c^n \det(A) \neq c \det(A)$.

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