SF 1684 Algebra and Geometry
Lecture 19

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Topics for Today

1. Quadratic Forms
2. Geometry of Quadratic Forms

Linear Forms

Recall that we up until now we have only been interested in equations of the form

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We have then discussed linear transformations and their geometry and how eigenvalues and eigenvectors play into the understanding of their geometry and their change of variables.

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A **quadratic form on** $\mathbb{R}^n$ is a polynomial in n variables where the total degree of each term is 2.For example:

$$Q(\vec{x}) = a_1 x_1^2 + a_2 x_2^2 + \cdots + a_n x^n + a_{n+1} x_1 x_2 + a_{n+2} x_1 x_3 + \cdots + a_* x_5 x_7 + \dots$$

cross terms.

Example of Quadratic Form

Consider the quadratic form

$$Q(\vec{x}) = x_1^2 + 4x_1x_2 + 3x_2^2$$

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$$= x_1(x_1 + 2x_2) + x_2(2x_1 + 2x_3)$$

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$$Q(\vec{x}) \quad = \vec{x}^T A \vec{x}$$

Quadratic Forms and Matrices

Theorem

For any quadratic form on $\mathbb{R}^n$, Q, you can find a square $n \times n$ matrix such that $Q(\vec{x}) = \vec{x}^T A\vec{x}$

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Then setting

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{pmatrix}$$

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Suppose Transformation is just a map $\mathbb{R}^n \to \mathbb{R}^m$

$$Q(\vec{x}) = a_{1,1}x_1x_1 + a_{1,2}x_1x_2 + \cdots + a_{1,n}x_1x_n + a_{2,1}x_2x_1 + a_{2,2}x_2x_2 + \ldots$$

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Exercise: expand $x^T A x$ to see that you indeed get this

We get $Q(\vec{x}) = \vec{x}^T A\vec{x}$.

$T_A(x) = A\vec{x}$

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We get $Q(\vec{x}) = \vec{x}^T A\vec{x}$. $\square$

Quadratic Forms and Symmetric Matrices

Note that in the above proof we wrote

$$Q(\vec{x}) = a_{1,1}x_1x_1 + a_{1,2}x_1x_2 + \cdots + a_{1,n}x_1x_n + a_{2,1}x_2x_1 + a_{2,2}x_2x_2 + \dots$$

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where we have just set $a'_{1,2} = \frac{1}{2}(a_{1,2} + a_{2,1})$ and so on.

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multiply by 2

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A' is symmetric $\rightarrow$

$$A' = \begin{pmatrix} a_{1,1} & a'_{1,2} & a'_{1,3} & \ldots & a'_{1,n} \\ a'_{1,2} & a_{2,2} & a'_{2,3} & \ldots & a'_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a'_{1,n} & a'_{2,n} & a'_{3,n} & \ldots & a_{n,n} \end{pmatrix}$$

Don't have multiplication by 2

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we see that A' is *symmetric* and $Q(\vec{x}) = \vec{x}^T A'\vec{x}$.

Concrete Example

Exercise

Explicitly write down the quadratic form for the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ and find a symmetric matrix A' that gives the same quadratic form.

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$$Q(\vec{x}) = \vec{x}^T A\vec{x}$$

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$$Q(\vec{x}) = \vec{x}^T(A\vec{x}) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \left(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right)$$

because A is $3x3$ we know $\vec{x} \in \mathbb{R}^3$

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$$Q(\vec{x}) = \vec{x}^T A \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \left(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right)$$

$$= 1x_1x_1 + 2x_1x_2 + 3x_1x_3 + 4x_1x_2 + 5x_2x_2 + 6x_2x_3 + 7x_3x_1 + 8x_3x_2 + 9x_3x_3$$

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$$= x_1^2 + 5x_2^2 + 9x_3^2 + 6x_1x_2 + 10x_1x_3 + 14x_2x_3$$

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$$Q(\vec{x}) = \vec{x}^T A\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \left(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right)$$

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$$= 1x_1^2 + 5x_2^2 + 9x_3^2 + 2(3x_1x_2) + 2(5x_1x_3) + 2(7x_2x_3)$$

Concrete Example Continued

Hence we see that if $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, then

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$$= \vec{x}^T \begin{pmatrix} 1 & 3 & 2 \\ 3 & 5 & 7 \\ 2 & 7 & 9 \end{pmatrix} \vec{x} = \vec{x}^T A' \vec{x}$$

The Quadratic Form of a Symmetric Matrix

Therefore, when we are talking about the matrix of a quadratic form we may always assume it is symmetric.

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Definition

Given an $n \times n$ symmetric matrix A, we define the **quadratic form associated with** A to be

$$Q_A(\vec{x}) = \vec{x}^T A \vec{x}$$

Simplest Quadratic Forms

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Simplest Quadratic Forms

The simplest quadratic forms will be the ones that are associated to the simplest matrices, which are diagonal matrices. Now, if

$$D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$$

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$$= d_1 x_1^2 + d_2 x_2^2 + \dots + d_n x_n^2$$

Principle Axes Theorem

Since our quadratic forms can always be associated with symmetric matrices, we can always *orthogonally* diagonalize these matrices.

A symmetric $\Rightarrow$ can P orthogonal & D diagonal

such that $A = P^T D P$

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orthonormal

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Now, we can view P as a change of basis operation. Hence, if we denote $\vec{y} = P\vec{x}$, this is essentially just looking at $\vec{x}$ is a different basis. Moreover, we get

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$\Rightarrow D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ where λ's are eigenvalues of A.

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where the λ_i are the diagonal entries of D, which are the *eigenvalues* of A.

Example

Exercise

Let $Q(\vec{x}) = x_1^2 - x_3^2 - 4x_1x_2 + 4x_2x_3$. Find a change of basis such that $Q(\vec{y}) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$.

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$\to 0 \, x_1 x_3$

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$$A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{pmatrix}$$

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$-2x_1x_2 \quad -2x_2x_1$

Suppose we put -4 that cell

$-4x_1x_2 \quad -4x_2x_1$

Now, we must orthogonally diagonalize A. Without showing the work, we get that the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = -3$ and $\lambda_2 = 3$ and that we can find an *orthonormal* basis of eigenvectors:

$$\vec{v}_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1/3 \\ -2/3 \\ 2/3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -2/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

Example 2

Hence,

$$A = P^T D P$$

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transpose

$$A = P^T D P = \begin{pmatrix} 2/3 & 1/3 & 2/3 \\ -1/3 & -2/3 & 2/3 \\ -2/3 & 2/3 & 2/3 \end{pmatrix} \begin{pmatrix} \overset{\lambda_1}{0} & 0 & 0 \\ 0 & \overset{\lambda_2}{-3} & 0 \\ 0 & 0 & \underset{\lambda_3}{3} \end{pmatrix} \begin{array}{c} \begin{pmatrix} 2/3 & -1/3 & -2/3 \\ 1/3 & -2/3 & 2/3 \\ 2/3 & 2/3 & 2/3 \end{pmatrix} \\ \begin{array}{ccc} v_1 & \quad v_2 & \quad v_3 \end{array} \end{array}$$

Example 2

Hence,

$$A = P^T DP = \begin{pmatrix} 2/3 & 1/3 & 2/3 \\ -1/3 & -2/3 & 2/3 \\ -2/3 & 2/3 & 2/3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2/3 & -1/3 & -2/3 \\ 1/3 & -2/3 & 2/3 \\ 2/3 & 2/3 & 2/3 \end{pmatrix}$$

and so

$$Q_A(\vec{x}) = \vec{x}^T A\vec{x}$$

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Hence,

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and so

$0y_1$

$$Q_A(\vec{x}) = \vec{x}^T A \vec{x} = \vec{x}^T P^T D P \vec{x} = (P\vec{x})^T D (P\vec{x}) = \vec{y}^T D \vec{y} = -3y_2^2 + 3y_3^2$$

where

$$\vec{y} = P\vec{x} = \begin{pmatrix} 2/3 & -1/3 & -2/3 \\ 1/3 & -2/3 & 2/3 \\ 2/3 & 2/3 & 2/3 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}x_1 - \frac{1}{3}x_2 - \frac{2}{3}x_3 \\ \frac{1}{3}x_1 - \frac{2}{3}x_2 + \frac{2}{3}x_3 \\ \frac{2}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3 \end{bmatrix} \begin{matrix} = y_1 \\ = y_1 \\ = y_3 \end{matrix}$$

Example 3

Confirm the fact that

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$$= -3\left(\frac{1}{3}x_1 - \frac{2}{3}x_2 + \frac{2}{3}x_3\right)^2 + 3\left(\frac{2}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3\right)^2$$

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$$= x_1^2 - x_3^2 - 4x_1x_2 + 4x_2x_3$$

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Exercise: do this expansion

$$= -3\left(\frac{1}{3}x_1 - \frac{2}{3}x_2 + \frac{2}{3}x_3\right)^2 + 3\left(\frac{2}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3\right)^2$$

$$= x_1^2 - x_3^2 - 4x_1x_2 + 4x_2x_3$$

$$= Q_A(\vec{x})$$

Geometry of Quadratic Forms

Much like how we wish to understand the solutions of $A\vec{x} = \vec{b}$ using geometry, we also would like to understand the solutions of $Q_A(\vec{x}) = k$ using geometry.

$Q_A(\vec{x}) \in \mathbb{R}$

Geometry of Quadratic Forms

Much like how we wish to understand the solutions of $A\vec{x} = \vec{b}$ using geometry, we also would like to understand the solutions of $Q_A(\vec{x}) = k$ using geometry. Let us first start with the simplest example:

Exercise

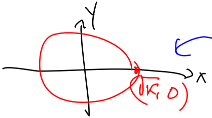
Geometrically explain the solutions to $Q_{I_2}(\vec{x}) = k$.

$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$Q_{I_2}(\vec{x}) = \vec{x}^T I_2 \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} \cdot \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2$

What are the solutions in $\mathbb{R}^2$ to $Q_{I_2}(\vec{x}) = x^2 + y^2 = k$?

The solutions geometrically form a circle centered at $(0,0)$ with radius $\sqrt{k}$



graph of $Q_{I_2}(\vec{x}) = k$

Geometry of a 2×2 Diagonal

Exercise

If $D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$ with $d_1, d_2 > 0$, geometrically explain the solutions to the quadratic form $Q_D(\vec{x}) = k$ with $k > 0$.

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We note that

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

Geometry of a 2×2 Diagonal

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If $D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$ with $d_1, d_2 > 0$, geometrically explain the solutions to the quadratic form $Q_D(\vec{x}) = k$ with $k > 0$.

We note that

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} = \begin{pmatrix} \sqrt{d_1} & 0 \\ 0 & \sqrt{d_2} \end{pmatrix} \begin{pmatrix} \sqrt{d_1} & 0 \\ 0 & \sqrt{d_2} \end{pmatrix}$$

Geometry of a 2×2 Diagonal

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Hence,

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Hence,

$$Q_D(\vec{x}) = \vec{x}^T D \vec{x} = \vec{x}^T B^T B \vec{x} = (B\vec{x})^T B \vec{x}$$

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Hence,

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Geometry of a 2×2 Diagonal 2

Hence, we can view the solutions to $Q_D(\vec{x}) = Q_{I_2}(B\vec{x}) = k$

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$\vec{y} = B\vec{x}$ $\quad$ $\vec{y}$ must lie on the circle of radius k.

Geometry of a 2×2 Diagonal 2

Hence, we can view the solutions to $Q_D(\vec{x}) = Q_{I_2}(B\vec{x}) = k$ as the set of $\vec{x}$ who, after the action of B, lie on the circle of radius k. So, what does the action of B do?

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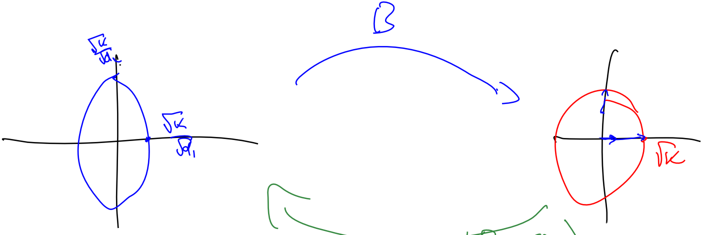
Hence, we can view the solutions to $Q_D(\vec{x}) = Q_{I_2}(B\vec{x}) = k$ as the set of $\vec{x}$ who, after the action of B, lie on the circle of radius k. So, what does the action of B do? Stretches the x-axis by $\sqrt{d_1}$ and the y-axis by $\sqrt{d_2}$.

$$B = \begin{pmatrix} \sqrt{d_1} & 0 \\ 0 & \sqrt{d_2} \end{pmatrix}$$

$\vec{x}$ $\qquad\qquad$ $\vec{y} = B\vec{x}$

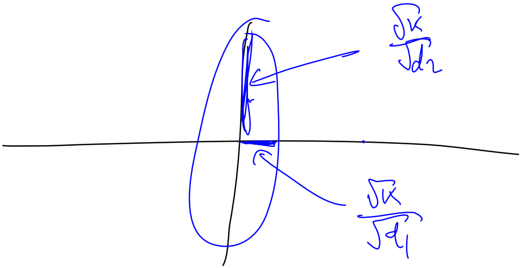


$$B^T = \begin{pmatrix} 1/\sqrt{d_1} & 0 \\ 0 & 1/\sqrt{d_2} \end{pmatrix}$$

Geometry of a 2×2 Diagonal 2

Hence, we can view the solutions to $Q_D(\vec{x}) = Q_{I_2}(B\vec{x}) = k$ as the set of $\vec{x}$ who, after the action of B, lie on the circle of radius k. So, what does the action of B do? Stretches the x-axis by $\sqrt{d_1}$ and the y-axis by $\sqrt{d_2}$.

Hence, the set of solutions to $Q_D(\vec{x}) = k$ is the ellipse whose x-radius is of length $\frac{\sqrt{k}}{\sqrt{d_1}}$ and whose y-radius is of length $\frac{\sqrt{k}}{\sqrt{d_2}}$.



Geometry of a 2×2 Diagonal 3

Exercise

Let $D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, geometrically explain the solutions to the quadratic form $Q_D(\vec{x}) = k$ and $k > 0$

Geometry of a 2×2 Diagonal 3

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We see that

$$Q_D(\vec{x}) = \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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We see that

$$Q_D(\vec{x}) = \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -x^2 + y^2$$

So then $Q_D(\vec{x}) = k \implies y^2 = x^2 + k$

Geometry of a 2 × 2 Diagonal 3

Exercise

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We see that

$y = x^2 + k$

$\implies$ parabola

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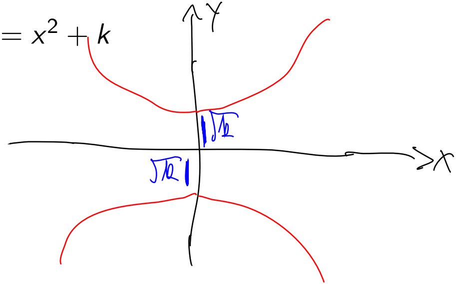
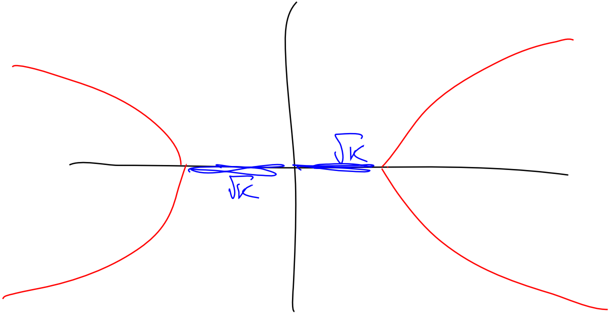


Geometry of a 2×2 Diagonal 4

Similarly if $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then $Q_D(\vec{x}) = k \implies x^2 = y^2 + k$ and we get a hyperbola.



Assuming $k > 0$

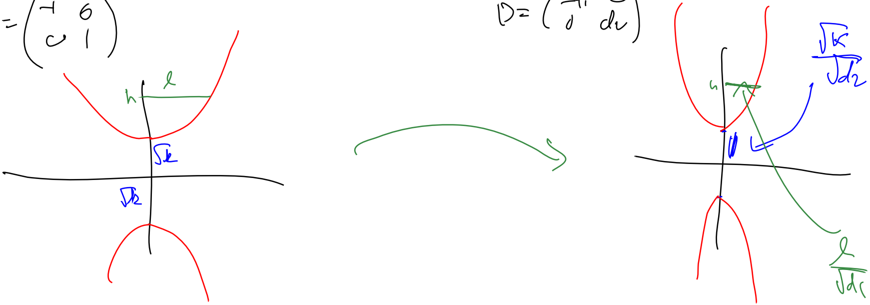
Geometry of a 2 × 2 Diagonal 4

Similarly if $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then $Q_D(\vec{x}) = k \implies x^2 = y^2 + k$ and we get a hyperbola.

Further, if $D = \begin{bmatrix} -d_1 & 0 \\ 0 & d_2 \end{bmatrix}$ or $\begin{bmatrix} d_1 & 0 \\ 0 & -d_2 \end{bmatrix}$ with $d_1, d_2 > 0$, then we get that $Q_D(\vec{x}) = k$ will be either a parabola or a hyperbola whose x-axis was stretched by a factor of $\frac{1}{\sqrt{d_1}}$ and y-axis was stretched by a factor of $\frac{1}{\sqrt{d_2}}$.

$D' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$D = \begin{pmatrix} -d_1 & 0 \\ 0 & d_2 \end{pmatrix}$



Finally, if $D = \begin{pmatrix} -d_1 & 0 \\ 0 & -d_2 \end{pmatrix}$ with $d_1, d_2 > 0$ then

always negative

$$Q_D(\vec{x}) = -d_1x^2 - d_2y^2 = k$$

has no solutions if $k > 0$ but is just the ellipse if $k < 0$.

$$-d_1x^2 - d_2y^2 = -2$$

$$\Rightarrow \quad d_1x^2 + d_2y^2 = 2$$

Finally, if $D = \begin{pmatrix} -d_1 & 0 \\ 0 & -d_2 \end{pmatrix}$ with $d_1, d_2 > 0$ then

$$Q_D(\vec{x}) = -d_1 x^2 - d_2 y^2 = k$$

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$>$

Moreover, if $D = \begin{pmatrix} -d_1 & 0 \\ 0 & d_2 \end{pmatrix}$ with $d_1, d_2 > 0$ then $\Rightarrow$ $d_1 x^2 - d_2 y^2 = k$

defines parabola

$$Q_D(\vec{x}) = -d_1 x^2 + d_2 y^2 = -k$$

$D' = \begin{pmatrix} d_1 & 0 \\ 0 & -d_2 \end{pmatrix} = -D$

is the same as $Q_{-D}(\vec{x}) = k$, and so would be a hyperbola.

defines a hyperbola

Finally, if $D = \begin{pmatrix} -d_1 & 0 \\ 0 & -d_2 \end{pmatrix}$ with $d_1, d_2 > 0$ then

$$Q_D(\vec{x}) = -d_1x^2 - d_2y^2 = k$$

has no solutions if $k \underset{>}{\cancel{<}} 0$ but is just the ellipse if $k < 0$.

Moreover, if $D = \begin{pmatrix} -d_1 & 0 \\ 0 & d_2 \end{pmatrix}$ with $d_1, d_2 > 0$ then

$$Q_D(\vec{x}) = -d_1x^2 + d_2y^2 = -k$$

is the same as $Q_{-D}(\vec{x}) = k$, and so would be a hyperbola.

Hence, we may always assume $k > 0$

Geometry of an Arbitrary 2×2

Exercise

If A is any symmetric 2×2 matrix, geometrically describe the solution $Q(\vec{x}) = k$, $k > 0$.

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$$Q_A(\vec{x}) = Q_D(P\vec{x}) = Q_D(\vec{y})$$

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Moreover, $\vec{y} = P\vec{x}$ can viewed as just an orthonormal change of basis. Thus, in the basis of eigenvectors of A, we know that Q_A will be an ellipse, parabola, or hyperbola depending on the properties of D.

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in the standard basis

That is, $Q_A(\vec{x})$ will be an ellipse, parabola or hyperbola stretched in the direction the eigenvectors of A.

Example

Exercise

Sketch the solutions to $Q_A(\vec{x}) = 36$ where $A = \begin{pmatrix} 5 & -2 \\ -2 & 8 \end{pmatrix}$.

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$$\lambda_1 = 4, \lambda_2 = 9, \vec{v}_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \vec{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

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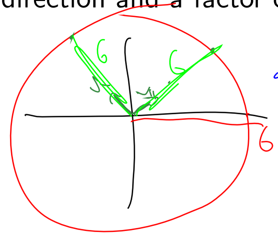
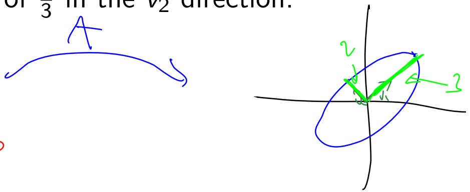
$\sqrt{\lambda_1}$ $\sqrt{36}$

Hence, it looks like a circle of radius 6 that has been "stretched" by $\frac{1}{2}$ in the $\vec{v}_1$ direction and a factor of $\frac{1}{3}$ in the $\vec{v}_2$ direction.





we call this v_1 & v_2 the principle axes of the ellipse.

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3. **indefinite** if $Q_A(\vec{x})$ has both positive and negative values

Definiteness and Eigenvalues

We see to understand the geometry of $Q_A(\vec{x})$ it is necessary to understand the geometry of $Q_D(\vec{x})$ which only depends on the eigenvalues of A.

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If A is a symmetric matrix then

→ ellipse

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   → no graph for $k>0$ (or ellipse for $k<0$)

2. *$Q_A(\vec{x})$ is negative definite if and only if all the eigenvalues of A are negative*
3. *$Q_A(\vec{x})$ is indefinite if and only if at least one eigenvalue is positive and at least one is negative*

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Note, if $Q_A(\vec{x})$ is negative definite, then $Q_{-A}(\vec{x})$ is positive definite so we may only consider positive definite and indefinite.

Positive Definiteness and Squares

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If A is a symmetric matrix, then the following statements are equivalent

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Positive Definiteness and Squares

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If A is a symmetric matrix, then the following statements are equivalent

1. *A is positive definite* $\longrightarrow$ all eigenvalues are positive
2. *There is a B such that $A = B^2$*

Positive Definiteness and Squares

Theorem

If A is a symmetric matrix, then the following statements are equivalent

1. *A is positive definite*
2. *There is a B such that $A = B^2$*
3. *There is an invertible matrix C such that $A = C^T C$*

$A = P^T D P$ all eigenvalues of positive

$D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} = \begin{pmatrix} \sqrt{d_1} & & \\ & \ddots & \\ & & \sqrt{d_n} \end{pmatrix} \begin{pmatrix} \sqrt{d_1} & & \\ & \ddots & \\ & & \sqrt{d_n} \end{pmatrix}$

$B = P^T \begin{pmatrix} \sqrt{d_1} & & \\ & \ddots & \\ & & \sqrt{d_n} \end{pmatrix} P$ satisfies $A = B^2$

Moreover B is symmetric

so $B^T = B$

So $B^2 = B^T B$

So setting $C = B$ we get $A = C^T C$

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Positive Definiteness and Squares

Tokeaway, if A is pos. def the $Q_A(x) = k$ geometrically looks like an n-dim ellipse.

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Definiteness and 2×2 Matrices.

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or parabola.