SF 1684 Algebra and Geometry
Lecture 17

Patrick Meisner

KTH Royal Institute of Technology

Topics for Today

1. Similar Matrices
2. Diagonalization
3. Eigenvalues and Diagonalizability

Similar Matrices

We have seen that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and B and B' are two different bases for $\mathbb{R}^n$, then we get two different matrices that define T in each of the bases

Similar Matrices

We have seen that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and B and B' are two different bases for $\mathbb{R}^n$, then we get two different matrices that define T in each of the bases, namely:

$$[T]_B \qquad \text{and} \qquad [T]_{B'}$$

Similar Matrices

We have seen that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and B and B' are two different bases for $\mathbb{R}^n$, then we get two different matrices that define T in each of the bases, namely:

$$[T]_B \qquad \text{and} \qquad [T]_{B'}$$

Further, we know that we can relate these two matrices through the change of basis matrix.

Similar Matrices

We have seen that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and B and B' are two different bases for $\mathbb{R}^n$, then we get two different matrices that define T in each of the bases, namely:

$$[T]_B \qquad \text{and} \qquad [T]_{B'}$$

Further, we know that we can relate these two matrices through the change of basis matrix. That is,

$$[T]_{B'} = P_{B \to B'}[T]_B P^{-1}_{B \to B'}$$

Similar Matrices

We have seen that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and B and B' are two different bases for $\mathbb{R}^n$, then we get two different matrices that define T in each of the bases, namely:

$$[T]_B \qquad \text{and} \qquad [T]_{B'}$$

Further, we know that we can relate these two matrices through the change of basis matrix. That is,

$$[T]_{B'} = P_{B\to B'}[T]_B P_{B\to B'}^{-1}$$

$[T]_{B'}$ is similar to $[T]_B$.

Definition

If A and C are square matrices of the same size, then we say that C **is similar to** A if there is an invertible matrix P such that $C = P^{-1}AP$.

First Properties of Similar Matrices

Theorem

1. *Two square matrices are similar if and only if there exists bases with respect to which the matrices represent the same linear transformation*
2. *Similar matrices have the same determinant*
3. *Similar matrices have the same trace*
4. *Similar matrices have the same nullity*
5. *Similar matrices have the same rank*

① If A & C are similar then $A = PCP^{-1}$ & then $P_{B \to B'}$ such that

$[T_A]_B = P\,[T_A]_{B'}\,P^{-1}$

② If $A = PCP^{-1}$ then $\det(A) = \det(PCP^{-1}) = \det(P)\det(C)\det(P^{-1})$

$= \det(P)\det(C)\frac{1}{\det P} = \det(C)$

③ $Tr(AB) = Tr(BA)$ for all matrices.

$Tr(A) = Tr(PCP^{-1})$

$= Tr(P^{-1}PC) = Tr(IC)$

$= Tr(C)$

Diagonalization

We saw in the previous slides that the matrices

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \qquad \text{and} \qquad C = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$

are similar

Diagonalization

We saw in the previous slides that the matrices

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \qquad \text{and} \qquad C = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$

are similar using the invertible matrix

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Diagonalization

We saw in the previous slides that the matrices

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \qquad \text{and} \qquad C = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$

are similar using the invertible matrix

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \text{that is, we showed } C = PAP^{-1}$$

Diagonalization

We saw in the previous slides that the matrices

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \qquad \text{and} \qquad C = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$

are similar using the invertible matrix

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \text{that is, we showed } C = PAP^{-1}$$

We were then able to use the fact that C was diagonal to easily interpret the linear transformations geometrically.

Diagonalization

We saw in the previous slides that the matrices

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \qquad \text{and} \qquad C = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$

are similar using the invertible matrix

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \text{that is, we showed } C = PAP^{-1}$$

We were then able to use the fact that C was diagonal to easily interpret the linear transformations geometrically.

Question (The Diagonalization Problem)

Given a square matrix A, does there exist an invertible matrix P for which $P^{-1}AP$ is a diagonal matrix, and if so, how does one find such a P?

Diagonalization

We saw in the previous slides that the matrices

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \qquad \text{and} \qquad C = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$

are similar using the invertible matrix

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \text{that is, we showed } C = PAP^{-1}$$

We were then able to use the fact that C was diagonal to easily interpret the linear transformations geometrically.

Question (The Diagonalization Problem)

Given a square matrix A, does there exist an invertible matrix P for which $P^{-1}AP$ is a diagonal matrix, and if so, how does one find such a P? If such a P exists, then A is said to be **diagonalizable** and P is said to **diagonalize** A.

Eigenvalues and Diagonalization

Recall, that we say that λ is an *eigenvalue* of a square matric A, if there exists a vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$.

Geometrically, this is saying that A acts by stretching by a factor of λ in the direction $\vec{v}$.

Eigenvalues and Diagonalization

Recall, that we say that λ is an *eigenvalue* of a square matric A, if there exists a vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$.

Theorem

If A is similar to the diagonal matrix

$$D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$$

Eigenvalues and Diagonalization

Recall, that we say that λ is an *eigenvalue* of a square matric A, if there exists a vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$.

Theorem

If A is similar to the diagonal matrix

$$D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$$

then $d_1, d_2, \dots, d_n$ are eigenvalues of A.

Eigenvalues and Diagonalization

Recall, that we say that λ is an *eigenvalue* of a square matric A, if there exists a vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$.

Theorem

If A is similar to the diagonal matrix

$$D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$$

then $d_1, d_2, \dots, d_n$ are eigenvalues of A.

Remark

Note that saying A is similar to a diagonal matrix is equivalent to saying that A is diagonalizable.

Proof

If A is similar to D. Then there exists an invertible P such that $A = PDP^{-1}$ $\quad D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$

Set $\vec{v}_i = P\vec{e}_i$ $\qquad e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th}$ position.

$$A\vec{v}_i = (PDP^{-1})(P\vec{e}_i) = PDP^{-1}P\vec{e}_i = PDI_n\vec{e}_i = PD\vec{e}_i$$

($P^{-1}P = I_n$)

$$D\vec{e}_i = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ d_i \\ \vdots \\ 0 \end{pmatrix} = d_i\vec{e}_i$$

d_i is a scalar as they the entries of D

$$A\vec{v}_i = PD\vec{e}_i = P(d_i\vec{e}_i) = d_i(P\vec{e}_i) = d_i\vec{v}_i$$

Thus I have found a vector $\vec{v}_i$ such that

$A\vec{v}_i = d_i\vec{v}_i$ & so d_i is an eigenvalue of A.

Eigenvectors and Diagonalization

Recall that we say $\vec{v}$ is an **eigenvector** of A if satisfies $A\vec{v} = \lambda\vec{v}$ for some eigenvalue λ.

Recall that we say $\vec{v}$ is an **eigenvector** of A if satisfies $A\vec{v} = \lambda\vec{v}$ for some eigenvalue λ.

Theorem

If a matrix A is diagonalizable and P is the invertible matrix that diagonalizes it, then the columns of P are eigenvectors of A.

Eigenvectors and Diagonalization

Recall that we say $\vec{v}$ is an **eigenvector** of A if satisfies $A\vec{v} = \lambda\vec{v}$ for some eigenvalue λ.

Theorem

If a matrix A is diagonalizable and P is the invertible matrix that diagonalizes it, then the columns of P are eigenvectors of A. Moreover, if

$$A = P \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix} P^{-1}$$

then the i^{th} column of P is an eigenvector of the eigenvalue d_i.

Proof

Since $A = PDP^{-1}$. We've already shown that if we take $\vec{v}_i = P\vec{e}_i$, then $A\vec{v}_i = d_i \vec{v}_i$.

In particular, this implies that $\vec{v}_i$ is an eigenvector of A that corresponds to the eigenvalue d_i.

$$V_i = P\vec{e}_i = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ \vdots & & & \vdots \\ p_{n1} & \cdots & & p_{nn} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} p_{1i} \\ p_{2i} \\ \vdots \\ p_{ni} \end{pmatrix} = i^{\text{th}} \text{ column of } P.$$

$\blacksquare$

Condition for Diagonalizable

Theorem

An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

($\Rightarrow$) If A is diagonalizable then $A = PDP^{-1}$ & all the columns of P are eigenvectors and there are n of them and since P is invertible they are linearly independent.

($\Leftarrow$) If A has n linearly independent eigenvectors then $\mathcal{B} = \{\vec{v}_1, \dots \vec{v}_n\}$ form a basis. $[A]_{\mathcal{B}}$ has the property that $[A]_{\mathcal{B}} [v_i]_{\mathcal{B}} = \lambda_i \cdot [v_i]_{\mathcal{B}}$

Exercise: show that this implies that $[A]_{\mathcal{B}}$ is diagonal.

How to Diagonalize

Corollary

If A has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and linearly independent eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ (where λ_i is the eigenvalue of $\vec{v}_i$),

How to Diagonalize

Corollary

If A has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and linearly independent eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ (where λ_i is the eigenvalue of $\vec{v}_i$), then if we set

$$P = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \end{pmatrix}$$

How to Diagonalize

Corollary

If A has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$*, and linearly independent eigenvectors* $\vec{v}_1, \ldots, \vec{v}_n$ *(where* λ_i *is the eigenvalue of* $\vec{v}_i$*), then if we set*

$$P = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \end{pmatrix}$$

we get

$$P^{-1}AP = D = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}$$

How to Diagonalize

Corollary

If A has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and linearly independent eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ (where λ_i is the eigenvalue of $\vec{v}_i$), then if we set

$$P = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \end{pmatrix}$$

we get

$$P^{-1}AP = D = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}$$

Hence, $A = PDP^{-1}$.

How to Diagonalize

Corollary

If A has eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, and linearly independent eigenvectors $\vec{v}_1, \dots, \vec{v}_n$ (where λ_i is the eigenvalue of $\vec{v}_i$), then if we set

$$P = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{pmatrix}$$

we get

$$P^{-1}AP = D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

Hence, $A = PDP^{-1}$.

We may then describe the linear transformation T_A geometrically by saying that it "stretches $\mathbb{R}^n$ in the direction of $\vec{v}_i$ by a factor of λ_i".

Example

Recall that an λ is an eigenvalue if and only if $\det(A - \lambda I_n) = 0$.

If λ is an eigenvalue we can find non-zero $\vec{v}$ s.t. $A\vec{v} = \lambda\vec{v}$

or $A\vec{v} - \lambda\vec{v} = 0$

or $(A - \lambda I_n)\vec{v} = 0$

equivalently $A - \lambda I_n$ is not invertible

equivalently $\det(A - \lambda I_n) = 0$

Example

Recall that an λ is an eigenvalue if and only if $\det(A - \lambda I_n) = 0$. Further, $\vec{v}$ is an eigenvector of the eigenvalue λ if only if $\vec{v}$ is in the null space of $A - \lambda I_n$.

$$A\vec{v} = \lambda\vec{v} \iff (A - \lambda I_n)\vec{v} = 0$$

$$\iff \vec{v} \in \text{null}(A - \lambda I_n)$$

Example

Recall that an λ is an eigenvalue if and only if $\det(A - \lambda I_n) = 0$. Further, $\vec{v}$ is an eigenvector of the eigenvalue λ if only if $\vec{v}$ is in the null space of $A - \lambda I_n$.

Exercise

Use these ideas to diagonalize $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

$$A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

Example

Recall that an λ is an eigenvalue if and only if $\det(A - \lambda I_n) = 0$. Further, $\vec{v}$ is an eigenvector of the eigenvalue λ if only if $\vec{v}$ is in the null space of $A - \lambda I_n$.

Exercise

Use these ideas to diagonalize $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

To find λ:

Example

Recall that an λ is an eigenvalue if and only if $\det(A - \lambda I_n) = 0$. Further, $\vec{v}$ is an eigenvector of the eigenvalue λ if only if $\vec{v}$ is in the null space of $A - \lambda I_n$.

Exercise

Use these ideas to diagonalize $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

To find λ:

$$\det(A - \lambda I_2)$$

Example

Recall that an λ is an eigenvalue if and only if $\det(A - \lambda I_n) = 0$. Further, $\vec{v}$ is an eigenvector of the eigenvalue λ if only if $\vec{v}$ is in the null space of $A - \lambda I_n$.

Exercise

Use these ideas to diagonalize $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

To find λ:

$$\det(A - \lambda I_2) = \det\left(\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

Example

Recall that an λ is an eigenvalue if and only if $\det(A - \lambda I_n) = 0$. Further, $\vec{v}$ is an eigenvector of the eigenvalue λ if only if $\vec{v}$ is in the null space of $A - \lambda I_n$.

Exercise

Use these ideas to diagonalize $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

To find λ:

$$\det(A - \lambda I_2) = \det\left(\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\left(\begin{pmatrix} 3 - \lambda & 2 \\ 2 & 3 - \lambda \end{pmatrix}\right)$$

Example

Recall that an λ is an eigenvalue if and only if $\det(A - \lambda I_n) = 0$. Further, $\vec{v}$ is an eigenvector of the eigenvalue λ if only if $\vec{v}$ is in the null space of $A - \lambda I_n$.

Exercise

Use these ideas to diagonalize $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

To find λ:

$$\det(A - \lambda I_2) = \det\left(\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\left(\begin{pmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{pmatrix}\right)$$

$$= (3-\lambda)^2 - 4$$

Example

Recall that an λ is an eigenvalue if and only if $\det(A - \lambda I_n) = 0$. Further, $\vec{v}$ is an eigenvector of the eigenvalue λ if only if $\vec{v}$ is in the null space of $A - \lambda I_n$.

Exercise

Use these ideas to diagonalize $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

To find λ:

$$\det(A - \lambda I_2) = \det\left(\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\left(\begin{pmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{pmatrix}\right)$$

$$= (3-\lambda)^2 - 4 = \lambda^2 - 6\lambda + 5$$

Example

Recall that an λ is an eigenvalue if and only if $\det(A - \lambda I_n) = 0$. Further, $\vec{v}$ is an eigenvector of the eigenvalue λ if only if $\vec{v}$ is in the null space of $A - \lambda I_n$.

Exercise

Use these ideas to diagonalize $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

To find λ:

$$\det(A - \lambda I_2) = \det\left(\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\left(\begin{pmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{pmatrix}\right)$$

$$= (3-\lambda)^2 - 4 = \lambda^2 - 6\lambda + 5 = (\lambda - 5)(\lambda - 1) = 0$$

Example

Recall that an λ is an eigenvalue if and only if $\det(A - \lambda I_n) = 0$. Further, $\vec{v}$ is an eigenvector of the eigenvalue λ if only if $\vec{v}$ is in the null space of $A - \lambda I_n$.

Exercise

Use these ideas to diagonalize $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

To find λ:

$$\det(A - \lambda I_2) = \det\left(\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\left(\begin{pmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{pmatrix}\right)$$

$$= (3-\lambda)^2 - 4 = \lambda^2 - 6\lambda + 5 = (\lambda - 5)(\lambda - 1) = 0$$

So the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 5$

Example 2

To find eigenvectors, we need to find null$(A - 1I_n)$ and null$(A - 5I_n)$

Example 2

To find eigenvectors, we need to find null$(A - I_n)$ and null$(A - 5I_n)$

$$A - I_n$$

Example 2

To find eigenvectors, we need to find null$(A - I_n)$ and null$(A - 5I_n)$

$$A - I_n = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example 2

To find eigenvectors, we need to find null$(A - I_n)$ and null$(A - 5I_n)$

$$A - I_n = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

Example 2

To find eigenvectors, we need to find null$(A - I_n)$ and null$(A - 5I_n)$

$$A - I_n = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

Example 2

To find eigenvectors, we need to find $\text{null}(A - I_n)$ and $\text{null}(A - 5I_n)$

$$A - I_n = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\implies \text{null}(A - I_2) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

Example 2

To find eigenvectors, we need to find $\text{null}(A - I_n)$ and $\text{null}(A - 5I_n)$

$$A - I_n = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\implies \text{null}(A - I_2) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

$A - 5I_n$

Example 2

To find eigenvectors, we need to find $\text{null}(A - I_n)$ and $\text{null}(A - 5I_n)$

$$A - I_n = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\implies \text{null}(A - I_2) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

$$A - 5I_n = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example 2

To find eigenvectors, we need to find null$(A - I_n)$ and null$(A - 5I_n)$

$$A - I_n = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\implies \text{null}(A - I_2) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

$$A - 5I_n = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$$

Example 2

To find eigenvectors, we need to find $\text{null}(A - I_n)$ and $\text{null}(A - 5I_n)$

$$A - I_n = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\implies \text{null}(A - I_2) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

$$A - 5I_n = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

Example 2

To find eigenvectors, we need to find null$(A - I_n)$ and null$(A - 5I_n)$

$$A - I_n = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\implies \text{null}(A - I_2) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

$$A - 5I_n = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$\implies \text{null}(A - 5I_2) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

Example 2

To find eigenvectors, we need to find $\text{null}(A - I_n)$ and $\text{null}(A - 5I_n)$

$$A - I_n = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\implies \text{null}(A - I_2) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

$$A - 5I_n = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$\implies \text{null}(A - 5I_2) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

Hence, we may conclude that

$$\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

Example 2

To find eigenvectors, we need to find $\text{null}(A - I_n)$ and $\text{null}(A - 5I_n)$

$$A - I_n = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\implies \text{null}(A - I_2) = \text{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{v_1} \right\}$$

$$A - 5I_n = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$\implies \text{null}(A - 5I_2) = \text{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{v_2} \right\}$$

Hence, we may conclude that

$$\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} = \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$$

Example 2

To find eigenvectors, we need to find $\text{null}(A - I_n)$ and $\text{null}(A - 5I_n)$

$$A - I_n = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\implies \text{null}(A - I_2) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

$$A - 5I_n = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$\implies \text{null}(A - 5I_2) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

Hence, we may conclude that

$$\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} = \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$

Example 2

To find eigenvectors, we need to find $\text{null}(A - I_n)$ and $\text{null}(A - 5I_n)$

$$A - I_n = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\implies \text{null}(A - I_2) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

$$A - 5I_n = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$\implies \text{null}(A - 5I_2) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

Hence, we may conclude that

$$\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1}$$

Example 2

To find eigenvectors, we need to find $\text{null}(A - I_n)$ and $\text{null}(A - 5I_n)$

$$A - I_n = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\implies \text{null}(A - I_2) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

$$A - 5I_n = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$\implies \text{null}(A - 5I_2) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

Hence, we may conclude that

$$\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1}$$

Permuting Eigenvalues

It is important that you are consistent with the eigenvalues and eigenvectors!

Permuting Eigenvalues

It is important that you are consistent with the eigenvalues and eigenvectors!

That is, the eigenvector of the first column must correspond with the first eigenvalue on the diagonal and so on.

Permuting Eigenvalues

It is important that you are consistent with the eigenvalues and eigenvectors!

That is, the eigenvector of the first column must correspond with the first eigenvalue on the diagonal and so on.

However, as long as you maintain consistency, you can rearrange as you want. That is,

Permuting Eigenvalues

It is important that you are consistent with the eigenvalues and eigenvectors!

That is, the eigenvector of the first column must correspond with the first eigenvalue on the diagonal and so on.

However, as long as you maintain consistency, you can rearrange as you want. That is,

$$\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1}$$

Permuting Eigenvalues

It is important that you are consistent with the eigenvalues and eigenvectors!

That is, the eigenvector of the first column must correspond with the first eigenvalue on the diagonal and so on.

However, as long as you maintain consistency, you can rearrange as you want. That is,

$$\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1}$$

but

$$\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$$

Exercise: do this multiplication & see that is correct.

Changing Eigenvectors

Moreover, the eigenvectors you choose have some freedom.

Changing Eigenvectors

Moreover, the eigenvectors you choose have some freedom. You may change them as long as you make sure they remain a basis!

Changing Eigenvectors

Moreover, the eigenvectors you choose have some freedom. You may change them as long as you make sure they remain a basis! That is, we have previously seen that

$$\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}^{-1}$$

Exercise: expand this.

Changing Eigenvectors

Moreover, the eigenvectors you choose have some freedom. You may change them as long as you make sure they remain a basis! That is, we have previously seen that

$$\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}^{-1}$$

This is fine since

$$\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$$

forms a basis for $\mathbb{R}^2$ where the first vector is an eigenvector of the eigenvalue 1 and the second vector is an eigenvector of the eigenvalue 5.

Changing Eigenvectors

Moreover, the eigenvectors you choose have some freedom. You may change them as long as you make sure they remain a basis! That is, we have previously seen that

$$\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}^{-1}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

This is fine since

$$\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$$

← orthonormal basis of eigenvectors

forms a basis for $\mathbb{R}^2$ where the first vector is an eigenvector of the eigenvalue 1 and the second vector is an eigenvector of the eigenvalue 5.

One reason we may want to consider this, somewhat more complicated basis, is that it is *orthonormal* whereas the one we found in the example was only orthogonal.

Not Diagonalizable

Exercise

Show that the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is **NOT** diagonalizable.

Let's try and diagonalize it by finding it's eigenvalues and eigenvectors:

Not Diagonalizable

Exercise

Show that the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is **NOT** diagonalizable.

Let's try and diagonalize it by finding it's eigenvalues and eigenvectors:

$$\det(A - \lambda I_2)$$

Not Diagonalizable

Exercise

Show that the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is **NOT** diagonalizable.

Let's try and diagonalize it by finding it's eigenvalues and eigenvectors:

$$\det(A - \lambda I_2) = \det\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

Not Diagonalizable

Exercise

Show that the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is **NOT** diagonalizable.

Let's try and diagonalize it by finding it's eigenvalues and eigenvectors:

$$\det(A - \lambda I_2) = \det\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\left(\begin{pmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{pmatrix}\right)$$

Not Diagonalizable

Exercise

Show that the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is **NOT** diagonalizable.

Let's try and diagonalize it by finding it's eigenvalues and eigenvectors:

$$\det(A - \lambda I_2) = \det\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\left(\begin{pmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{pmatrix}\right)$$

$$= (1-\lambda)^2 - 0 \times 1$$

Not Diagonalizable

Exercise

Show that the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is **NOT** diagonalizable.

Let's try and diagonalize it by finding it's eigenvalues and eigenvectors:

$$\det(A - \lambda I_2) = \det\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\left(\begin{pmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{pmatrix}\right)$$

$$= (1-\lambda)^2 - 0 \times 1 = (\lambda - 1)^2 = 0$$

Not Diagonalizable

Exercise

Show that the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is **NOT** diagonalizable.

Let's try and diagonalize it by finding it's eigenvalues and eigenvectors:

$$\det(A - \lambda I_2) = \det\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\left(\begin{pmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{pmatrix}\right)$$

$$= (1-\lambda)^2 - 0 \times 1 = (\lambda - 1)^2 = 0$$

So we only get one eigenvalue: $\lambda_1 = 1$.

Not Diagonalizable

Exercise

Show that the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is **NOT** diagonalizable.

Let's try and diagonalize it by finding it's eigenvalues and eigenvectors:

$$\det(A - \lambda I_2) = \det\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\left(\begin{pmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{pmatrix}\right)$$

$$= (1-\lambda)^2 - 0 \times 1 = (\lambda - 1)^2 = 0$$

So we only get one eigenvalue: $\lambda_1 = 1$.

$A - I_n$

Not Diagonalizable

Exercise

Show that the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is **NOT** diagonalizable.

Let's try and diagonalize it by finding it's eigenvalues and eigenvectors:

$$\det(A - \lambda I_2) = \det\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\left(\begin{pmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{pmatrix}\right)$$

$$= (1-\lambda)^2 - 0 \times 1 = (\lambda - 1)^2 = 0$$

So we only get one eigenvalue: $\lambda_1 = 1$.

$$A - I_n = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Not Diagonalizable

Exercise

Show that the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is **NOT** diagonalizable.

Let's try and diagonalize it by finding it's eigenvalues and eigenvectors:

$$\det(A - \lambda I_2) = \det\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\left(\begin{pmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{pmatrix}\right)$$

$$= (1-\lambda)^2 - 0 \times 1 = (\lambda - 1)^2 = 0$$

So we only get one eigenvalue: $\lambda_1 = 1$.

$$A - I_n = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Not Diagonalizable

Exercise

Show that the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is **NOT** diagonalizable.

Let's try and diagonalize it by finding it's eigenvalues and eigenvectors:

$$\det(A - \lambda I_2) = \det\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\left(\begin{pmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{pmatrix}\right)$$

$$= (1-\lambda)^2 - 0 \times 1 = (\lambda - 1)^2 = 0$$

So we only get one eigenvalue: $\lambda_1 = 1$.

$$A - I_n = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \implies \text{null}(A - I_2) = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

Not Diagonalizable

Exercise

Show that the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is **NOT** diagonalizable.

Let's try and diagonalize it by finding it's eigenvalues and eigenvectors:

$$\det(A - \lambda I_2) = \det\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\left(\begin{pmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{pmatrix}\right)$$

$$= (1-\lambda)^2 - 0 \times 1 = (\lambda - 1)^2 = 0$$

$V = \begin{bmatrix} 0 \\ 1 \end{bmatrix} t$

So we only get one eigenvalue: $\lambda_1 = 1$.

$$A - I_n = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \implies \text{null}(A - I_2) = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

Hence, we only get **ONE** linearly independent eigenvector instead of the **TWO** we need.

3×3 Example

The issue in the above example is NOT in the fact that we had fewer eigen*values* than we had dimensions.

3×3 Example

The issue in the above example is NOT in the fact that we had fewer eigen*values* than we had dimensions. For instance, the matrix

$$A = \begin{pmatrix} 1/2 & -1 & 1/2 \\ 0 & 1 & 0 \\ -3/2 & -3 & 5/2 \end{pmatrix}$$

3×3 Example

The issue in the above example is NOT in the fact that we had fewer eigen*values* than we had dimensions. For instance, the matrix

$$A = \begin{pmatrix} 1/2 & -1 & 1/2 \\ 0 & 1 & 0 \\ -3/2 & -3 & 5/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix}^{-1}$$

$P \qquad D \qquad P^{-1}$

3×3 Example

The issue in the above example is NOT in the fact that we had fewer eigen*values* than we had dimensions. For instance, the matrix

$$A = \begin{pmatrix} 1/2 & -1 & 1/2 \\ 0 & 1 & 0 \\ -3/2 & -3 & 5/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix}^{-1}$$

And so we see that A is diagonalizable yet has only two eigenvalues 1 and 2.

3×3 Example

The issue in the above example is NOT in the fact that we had fewer eigen*values* than we had dimensions. For instance, the matrix

$$A = \begin{pmatrix} 1/2 & -1 & 1/2 \\ 0 & 1 & 0 \\ -3/2 & -3 & 5/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix}^{-1}$$

And so we see that A is diagonalizable yet has only two eigenvalues 1 and 2. Moreover, we can read off the eigenvectors that correspond to 1 and 2 from the invertible matrix.

3×3 Example

The issue in the above example is NOT in the fact that we had fewer eigen*values* than we had dimensions. For instance, the matrix

$$A = \begin{pmatrix} 1/2 & -1 & 1/2 \\ 0 & 1 & 0 \\ -3/2 & -3 & 5/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix}^{-1}$$

And so we see that A is diagonalizable yet has only two eigenvalues 1 and 2. Moreover, we can read off the eigenvectors that correspond to 1 and 2 from the invertible matrix. That is, eigenvector with eigenvalues 1 and 2 are

$$\lambda = 1 \implies \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

3 × 3 Example

The issue in the above example is NOT in the fact that we had fewer eigen*values* than we had dimensions. For instance, the matrix

$$A = \begin{pmatrix} 1/2 & -1 & 1/2 \\ 0 & 1 & 0 \\ -3/2 & -3 & 5/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix}^{-1}$$

And so we see that A is diagonalizable yet has only two eigenvalues 1 and 2. Moreover, we can read off the eigenvectors that correspond to 1 and 2 from the invertible matrix. That is, eigenvector with eigenvalues 1 and 2 are

$$\lambda = 1 \implies \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \qquad \lambda = 2 \implies \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

form a basis

Eigenspaces

Hence, the issue with $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ came from the fact that the eigenvalue only corresponded to 1 linearly independent eigenvectors instead of the two we need.

Eigenspaces

Hence, the issue with $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ came from the fact that the eigenvalue only corresponded to 1 linearly independent eigenvectors instead of the two we need. This leads us to discuss the **eigenspaces**.

Eigenspaces

Hence, the issue with $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ came from the fact that the eigenvalue only corresponded to 1 linearly independent eigenvectors instead of the two we need. This leads us to discuss the **eigenspaces**.

Definition

If A is a matrix and λ is an eigenvalue of A, then we define the **eigenspace of** λ, denote E_λ, to be all the vectors $\vec{v}$ such that $\vec{v}$ is an eigenvector with eigenvalue λ.

Eigenspaces

Hence, the issue with $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ came from the fact that the eigenvalue only corresponded to 1 linearly independent eigenvectors instead of the two we need. This leads us to discuss the **eigenspaces**.

Definition

If A is a matrix and λ is an eigenvalue of A, then we define the **eigenspace of** λ, denote E_λ, to be all the vectors $\vec{v}$ such that $\vec{v}$ is an eigenvector with eigenvalue λ.Equivalently

$$E_\lambda = \{\vec{v} : A\vec{v} = \lambda\vec{v}\}$$

Eigenspaces

Hence, the issue with $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ came from the fact that the eigenvalue only corresponded to 1 linearly independent eigenvectors instead of the two we need. This leads us to discuss the **eigenspaces**.

Definition

If A is a matrix and λ is an eigenvalue of A, then we define the **eigenspace of** λ, denote E_λ, to be all the vectors $\vec{v}$ such that $\vec{v}$ is an eigenvector with eigenvalue λ.Equivalently

$$E_\lambda = \{\vec{v} : A\vec{v} = \lambda\vec{v}\}$$

or

$$E_\lambda = \mathrm{null}(A - \lambda I_n).$$

Distinct Eigenspaces are Linearly Independent

Theorem

Let A be an $n \times n$ matrix and let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues. Then if $\vec{v}_i \in E_{\lambda_i}$ for $i = 1, \ldots, k$, then the set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent.

all eigenvectors
all lie is different eigenspaces.
That is, they correspond to different eigenvalues.

Distinct Eigenspaces are Linearly Independent

Theorem

Let A be an $n \times n$ matrix and let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues. Then if $\vec{v}_i \in E_{\lambda_i}$ for $i = 1, \ldots, k$, then the set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent.

Sketch of Proof.

In the case $k = 2$, if $\vec{v}_1$ and $\vec{v}_2$ were linearly dependent, then $\vec{v}_1 = c\vec{v}_2$ for some c.

Distinct Eigenspaces are Linearly Independent

Theorem

Let A be an $n \times n$ matrix and let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues. Then if $\vec{v}_i \in E_{\lambda_i}$ for $i = 1, \ldots, k$, then the set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent.

Sketch of Proof.

In the case $k = 2$, if $\vec{v}_1$ and $\vec{v}_2$ were linearly dependent, then $\vec{v}_1 = c\vec{v}_2$ for some c.Hence,

$$\lambda_1 \vec{v}_1 = A\vec{v}_1$$

↑ because v_1 is a eigenvector of A with eigenvalue λ_1

Distinct Eigenspaces are Linearly Independent

Theorem

Let A be an $n \times n$ matrix and let $\lambda_1, \dots, \lambda_k$ be distinct eigenvalues. Then if $\vec{v}_i \in E_{\lambda_i}$ for $i = 1, \dots, k$, then the set $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent.

Sketch of Proof.

In the case $k = 2$, if $\vec{v}_1$ and $\vec{v}_2$ were linearly dependent, then $\vec{v}_1 = c\vec{v}_2$ for some c.Hence,

$$\lambda_1 \vec{v}_1 = A\vec{v}_1 = A(c\vec{v}_2)$$

↑ by assumption of linear dependence.

Distinct Eigenspaces are Linearly Independent

Theorem

Let A be an $n \times n$ matrix and let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues. Then if $\vec{v}_i \in E_{\lambda_i}$ for $i = 1, \ldots, k$, then the set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent.

Sketch of Proof.

In the case $k = 2$, if $\vec{v}_1$ and $\vec{v}_2$ were linearly dependent, then $\vec{v}_1 = c\vec{v}_2$ for some c.Hence,

$$\lambda_1 \vec{v}_1 = A\vec{v}_1 = A(c\vec{v}_2) = c(A\vec{v}_2)$$

Distinct Eigenspaces are Linearly Independent

Theorem

Let A be an $n \times n$ matrix and let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues. Then if $\vec{v}_i \in E_{\lambda_i}$ for $i = 1, \ldots, k$, then the set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent.

Sketch of Proof.

In the case $k = 2$, if $\vec{v}_1$ and $\vec{v}_2$ were linearly dependent, then $\vec{v}_1 = c\vec{v}_2$ for some c.Hence,

$$\lambda_1 \vec{v}_1 = A\vec{v}_1 = A(c\vec{v}_2) = c(A\vec{v}_2) = c(\lambda_2 \vec{v}_2)$$

↑

because v_2 is an eigenvector with eigenvalue λ_2.

Distinct Eigenspaces are Linearly Independent

Theorem

Let A be an $n \times n$ matrix and let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues. Then if $\vec{v}_i \in E_{\lambda_i}$ for $i = 1, \ldots, k$, then the set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent.

Sketch of Proof.

In the case $k = 2$, if $\vec{v}_1$ and $\vec{v}_2$ were linearly dependent, then $\vec{v}_1 = c\vec{v}_2$ for some c.Hence,

$$\lambda_1 \vec{v}_1 = A\vec{v}_1 = A(c\vec{v}_2) = c(A\vec{v}_2) = c(\lambda_2 \vec{v}_2) = \lambda_2(c\vec{v}_2)$$

Distinct Eigenspaces are Linearly Independent

Theorem

Let A be an $n \times n$ matrix and let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues. Then if $\vec{v}_i \in E_{\lambda_i}$ for $i = 1, \ldots, k$, then the set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent.

Sketch of Proof.

In the case $k = 2$, if $\vec{v}_1$ and $\vec{v}_2$ were linearly dependent, then $\vec{v}_1 = c\vec{v}_2$ for some c.Hence,

$$\lambda_1 \vec{v}_1 = A\vec{v}_1 = A(c\vec{v}_2) = c(A\vec{v}_2) = c(\lambda_2 \vec{v}_2) = \lambda_2(c\vec{v}_2) = \lambda_2 \vec{v}_1$$

↑ by previous assumption of linear dependence

And so, it would have to be that $\lambda_1 = \lambda_2$

Distinct Eigenspaces are Linearly Independent

Theorem

Let A be an $n \times n$ matrix and let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues. Then if $\vec{v}_i \in E_{\lambda_i}$ for $i = 1, \ldots, k$, then the set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent.

Sketch of Proof.

In the case $k = 2$, if $\vec{v}_1$ and $\vec{v}_2$ were linearly dependent, then $\vec{v}_1 = c\vec{v}_2$ for some c.Hence,

$$\lambda_1\vec{v}_1 = A\vec{v}_1 = A(c\vec{v}_2) = c(A\vec{v}_2) = c(\lambda_2\vec{v}_2) = \lambda_2(c\vec{v}_2) = \lambda_2\vec{v}_1$$

And so, it would have to be that $\lambda_1 = \lambda_2$, which contradicts the assumption that the λ_i were distinct. □

Corollary

Corollary

If an $n \times n$ matrix A has n distinct eigenvalues then it is diagonalizable.

If A has n distinct eigenvalues $\lambda_1 \ldots \lambda_n$.

The let $v_1 \ldots v_n$ be eigenvectors that correspond to $\lambda_1 \ldots \lambda_n$, respectively. Because the λ_i are distinct $\{v_1 \ldots v_n\}$ are linearly independent and so A is diagonalizable.

Corollary

Corollary

If an $n \times n$ matrix A has n distinct eigenvalues then it is diagonalizable.

Proof.

Let $\lambda_1, \ldots, \lambda_n$ be the n distinct eigenvalues of A.

Corollary

Corollary

If an $n \times n$ matrix A has n distinct eigenvalues then it is diagonalizable.

Proof.

Let $\lambda_1, \ldots, \lambda_n$ be the n distinct eigenvalues of A. Let $\vec{v}_1, \ldots, \vec{v}_n$ be any set of vectors such that $\vec{v}_i \in E_{\lambda_i}$ for $i = 1, \dot{;} n$.

Corollary

Corollary

If an $n \times n$ matrix A has n distinct eigenvalues then it is diagonalizable.

Proof.

Let $\lambda_1, \ldots, \lambda_n$ be the n distinct eigenvalues of A. Let $\vec{v}_1, \ldots, \vec{v}_n$ be any set of vectors such that $\vec{v}_i \in E_{\lambda_i}$ for $i = 1, \dot{,} n$. Then

$$\{\vec{v}_1, \ldots, \vec{v}_n\}$$

is a set of n linearly independent eigenvectors and so A is diagonalizable.

Corollary

Corollary

If an $n \times n$ matrix A has n distinct eigenvalues then it is diagonalizable.

Proof.

Let $\lambda_1, \dots, \lambda_n$ be the n distinct eigenvalues of A. Let $\vec{v}_1, \dots, \vec{v}_n$ be any set of vectors such that $\vec{v}_i \in E_{\lambda_i}$ for $i = 1, \dot{,} n$. Then

$$\{\vec{v}_1, \dots, \vec{v}_n\}$$

is a set of n linearly independent eigenvectors and so A is diagonalizable. In particular:

$$A = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{pmatrix}^{-1}$$

Geometric Multiplicity

Definition

If A is a matrix and λ is an eigenvalue, then we define the **geometric multiplicity** of λ to be the dimension of its eigenspace E_λ.

$\dim(E_\lambda) =$ # of linearly independent eigenvectors that correspond to λ.

Geometric Multiplicity

Definition

If A is a matrix and λ is an eigenvalue, then we define the **geometric multiplicity** of λ to be the dimension of its eigenspace E_λ.

Theorem

A $n \times n$ matrix A is diagonalizable if and only if the sum of the geometric multiplicities of its eigenvalues is n.

Geometric Multiplicity

Definition

If A is a matrix and λ is an eigenvalue, then we define the **geometric multiplicity** of λ to be the dimension of its eigenspace E_λ.

Theorem

A $n \times n$ matrix A is diagonalizable if and only if the sum of the geometric multiplicities of its eigenvalues is n.

Sketch of Proof.

Let $\lambda_1, \ldots, \lambda_k$ be the set of distinct eigenvalues.

Geometric Multiplicity

Definition

If A is a matrix and λ is an eigenvalue, then we define the **geometric multiplicity** of λ to be the dimension of its eigenspace E_λ.

Theorem

A $n \times n$ matrix A is diagonalizable if and only if the sum of the geometric multiplicities of its eigenvalues is n.

Sketch of Proof.

Let $\lambda_1, \ldots, \lambda_k$ be the set of distinct eigenvalues. Let g_i be the geometric multiplicity of λ_i.

$g_i = \dim(E_{\lambda_i}) \Leftrightarrow$ can find a basis of E_{λ_i} with g_i vectors.

Geometric Multiplicity

Definition

If A is a matrix and λ is an eigenvalue, then we define the **geometric multiplicity** of λ to be the dimension of its eigenspace E_λ.

Theorem

A $n \times n$ matrix A is diagonalizable if and only if the sum of the geometric multiplicities of its eigenvalues is n.

Sketch of Proof.

Let $\lambda_1, \ldots, \lambda_k$ be the set of distinct eigenvalues. Let g_i be the geometric multiplicity of λ_i. Then we can find a basis for each eigenspace E_{λ_i} as $E_{\lambda_i} = \text{span}\{\vec{v}_{i,1}, \vec{v}_{i,2}, \ldots, \vec{v}_{i,g_i}\}$

Geometric Multiplicity

Definition

If A is a matrix and λ is an eigenvalue, then we define the **geometric multiplicity** of λ to be the dimension of its eigenspace E_λ.

Theorem

A $n \times n$ matrix A is diagonalizable if and only if the sum of the geometric multiplicities of its eigenvalues is n.

Sketch of Proof.

Let $\lambda_1, \ldots, \lambda_k$ be the set of distinct eigenvalues. Let g_i be the geometric multiplicity of λ_i. Then we can find a basis for each eigenspace E_{λ_i} as $E_{\lambda_i} = \text{span}\{\vec{v}_{i,1}, \vec{v}_{i,2}, \ldots, \vec{v}_{i,g_i}\}$ lin ind

Then the set of vectors $\{\vec{v}_{1,1}, \vec{v}_{1,2}, \ldots, \vec{v}_{1,g_1}, \vec{v}_{2,1}, \ldots, \vec{v}_{k,g_k}\}$ is the largest *linearly independent* set of eigen~~values~~ vectors. Requires a little more work.

Geometric Multiplicity

Definition

If A is a matrix and λ is an eigenvalue, then we define the **geometric multiplicity** of λ to be the dimension of its eigenspace E_λ.

Theorem

A $n \times n$ matrix A is diagonalizable if and only if the sum of the geometric multiplicities of its eigenvalues is n.

Sketch of Proof.

Let $\lambda_1, \dots, \lambda_k$ be the set of distinct eigenvalues. Let g_i be the geometric multiplicity of λ_i. Then we can find a basis for each eigenspace E_{λ_i} as $E_{\lambda_i} = \text{span}\{\vec{v}_{i,1}, \vec{v}_{i,2}, \dots, \vec{v}_{i,g_i}\}$

Then the set of vectors $\{\vec{v}_{1,1}, \vec{v}_{1,2}, \dots, \vec{v}_{1,g_1}, \vec{v}_{2,1}, \dots, \vec{v}_{k,g_k}\}$ is the largest *linearly independent* set of eigen~~values~~s. vectors. Hence, A is diagonalizable if and only if $g_1 + g_2 + \cdots + g_k = n$. □

Characteristic Polynomial

Recall that the λ is an eigenvalue of A if and only if $\det(A - \lambda I_n) = 0$.

Characteristic Polynomial

Recall that the λ is an eigenvalue of A if and only if $\det(A - \lambda I_n) = 0$.If we view λ as a variable then we see that $\det(A - \lambda I_n)$ will be a polynomial of degree n.

Characteristic Polynomial

Recall that the λ is an eigenvalue of A if and only if $\det(A - \lambda I_n) = 0$.If we view λ as a variable then we see that $\det(A - \lambda I_n)$ will be a polynomial of degree n.

Definition

The polynomial given by $\det(A - tI_n)$ is called the **characteristic polynomial of** A.

t is a variable

Characteristic Polynomial

Recall that the λ is an eigenvalue of A if and only if $\det(A - \lambda I_n) = 0$.If we view λ as a variable then we see that $\det(A - \lambda I_n)$ will be a polynomial of degree n.

Definition

The polynomial given by $\det(A - tI_n)$ is called the **characteristic polynomial of** A. Moreover, we see that λ is an eigenvalue of A if and only if it is a root of the characteristic polynomial of A.

Characteristic Polynomial

Recall that the λ is an eigenvalue of A if and only if $\det(A - \lambda I_n) = 0$.If we view λ as a variable then we see that $\det(A - \lambda I_n)$ will be a polynomial of degree n.

Definition

The polynomial given by $\det(A - tI_n)$ is called the **characteristic polynomial of** A. Moreover, we see that λ is an eigenvalue of A if and only if it is a root of the characteristic polynomial of A.

We know that if $r_1, \ldots, r_n$ are the root of any polynomial $P(t)$, then we can write $P(t) = (t - r_1)(t - r_2) \cdots (t - r_n)$

Characteristic Polynomial

Recall that the λ is an eigenvalue of A if and only if $\det(A - \lambda I_n) = 0$.If we view λ as a variable then we see that $\det(A - \lambda I_n)$ will be a polynomial of degree n.

Definition

The polynomial given by $\det(A - tI_n)$ is called the **characteristic polynomial of** A. Moreover, we see that λ is an eigenvalue of A if and only if it is a root of the characteristic polynomial of A.

We know that if $r_1, \ldots, r_n$ are the root of any polynomial $P(t)$, then we can write $P(t) = (t - r_1)(t - r_2) \cdots (t - r_n)$Of course, the roots $r_1, r_2, \ldots, r_n$ may not be distinct.

Characteristic Polynomial

Recall that the λ is an eigenvalue of A if and only if $\det(A - \lambda I_n) = 0$.If we view λ as a variable then we see that $\det(A - \lambda I_n)$ will be a polynomial of degree n.

Definition

The polynomial given by $\det(A - tI_n)$ is called the **characteristic polynomial of** A. Moreover, we see that λ is an eigenvalue of A if and only if it is a root of the characteristic polynomial of A.

We know that if $r_1, \ldots, r_n$ are the root of any polynomial $P(t)$, then we can write $P(t) = (t - r_1)(t - r_2) \cdots (t - r_n)$Of course, the roots $r_1, r_2, \ldots, r_n$ may not be distinct. Hence, for any root r, we define the **multiplicity** of it to be the number of times it appears on the list of r_i.

Characteristic Polynomial

Recall that the λ is an eigenvalue of A if and only if $\det(A - \lambda I_n) = 0$.If we view λ as a variable then we see that $\det(A - \lambda I_n)$ will be a polynomial of degree n.

Definition

The polynomial given by $\det(A - tI_n)$ is called the **characteristic polynomial of** A. Moreover, we see that λ is an eigenvalue of A if and only if it is a root of the characteristic polynomial of A.

We know that if $r_1, \ldots, r_n$ are the root of any polynomial $P(t)$, then we can write $P(t) = (t - r_1)(t - r_2) \cdots (t - r_n)$Of course, the roots $r_1, r_2, \ldots, r_n$ may not be distinct. Hence, for any root r, we define the **multiplicity** of it to be the number of times it appears on the list of r_i. We can extend this to eigenvalues.

Algebraic Multiplicity

Definition

Let A be a matrix and let λ be an eigenvalue of A. Then we define the **algebraic multiplicity of** λ to be the multiplicity of λ as a root of the characteristic polynomial.

Algebraic Multiplicity

Definition

Let A be a matrix and let λ be an eigenvalue of A. Then we define the **algebraic multiplicity of** λ to be the multiplicity of λ as a root of the characteristic polynomial.

Note: the *algebraic* multiplicity of an eigenvalue λ is not to be confused with the *geometric* multiplicity of λ!

Algebraic Multiplicity

Definition

Let A be a matrix and let λ be an eigenvalue of A. Then we define the **algebraic multiplicity of** λ to be the multiplicity of λ as a root of the characteristic polynomial.

Note: the *algebraic* multiplicity of an eigenvalue λ is not to be confused with the *geometric* multiplicity of λ! While these two concepts are very much related, they are indeed different things.

Algebraic Multiplicity

Definition

Let A be a matrix and let λ be an eigenvalue of A. Then we define the **algebraic multiplicity of** λ to be the multiplicity of λ as a root of the characteristic polynomial.

Note: the *algebraic* multiplicity of an eigenvalue λ is not to be confused with the *geometric* multiplicity of λ! While these two concepts are very much related, they are indeed different things.

Theorem

Let A be a matrix and let $\lambda_1, \ldots, \lambda_k$ be the set of distinct eigenvalues of A. Let a_i be the algebraic multiplicity of λ_i for $i = 1, \ldots, k$. Then

$$a_1 + a_2 + \cdots + a_k = n$$

Relating Algebraic and Geometric Multiplicities

Theorem

Let A be a matrix and let $\lambda_1, \ldots, \lambda_k$ be a set of distinct eigenvalues of A. Let $a_1, \ldots, a_k$ and $g_1, \ldots, g_k$ be the algebraic and geometric multiplicities of A. Then

Relating Algebraic and Geometric Multiplicities

Theorem

Let A be a matrix and let $\lambda_1, \ldots, \lambda_k$ be a set of distinct eigenvalues of A. Let $a_1, \ldots, a_k$ and $g_1, \ldots, g_k$ be the algebraic and geometric multiplicities of A. Then

1. $1 \leq g_i \leq a_i$ *for all* $i = 1, \ldots, k$

Relating Algebraic and Geometric Multiplicities

Big Theorem

Theorem

Let A be a matrix and let $\lambda_1, \ldots, \lambda_k$ be a set of distinct eigenvalues of A. Let $a_1, \ldots, a_k$ and $g_1, \ldots, g_k$ be the algebraic and geometric multiplicities of A. Then

1. $1 \leq g_i \leq a_i$ *for all* $i = 1, \ldots, k$
2. *A is diagonalizable if and only if $a_i = g_i$ for all $i = 1, \ldots, k$.*

(2) A is diagonalizable iff $g_1 + \cdots + g_k = n$

$g_1 + \cdots + g_k \leq a_1 + \cdots + a_k = n$ so this equality can happen if inequality is also equality.

Rundown of Terminology in Examples

If $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$,

Rundown of Terminology in Examples

If $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$, then the characteristic polynomial is

$$\det(A - tI) = (t - 1)(t - 5)$$

Rundown of Terminology in Examples

If $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$, then the characteristic polynomial is

$$\det(A - tI) = (t - 1)(t - 5)$$

The eigenvalues are 1 and 5.

Rundown of Terminology in Examples

If $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$, then the characteristic polynomial is

$$\det(A - tI) = (t-1)^1(t-5)^1$$

The eigenvalues are 1 and 5. The arithmetic multiplicity of 1 is 1 and the arithmetic multiplicity of 5 is 1.

Rundown of Terminology in Examples

If $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$, then the characteristic polynomial is

$$\det(A - tI) = (t-1)(t-5)$$

The eigenvalues are 1 and 5. The arithmetic multiplicity of 1 is 1 and the arithmetic multiplicity of 5 is 1. The eigenspaces are

$$E_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \quad E_5 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

Rundown of Terminology in Examples

If $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$, then the characteristic polynomial is

$$\det(A - tI) = (t-1)(t-5)$$

The eigenvalues are 1 and 5. The arithmetic multiplicity of 1 is 1 and the arithmetic multiplicity of 5 is 1. The eigenspaces are

$\dim(E_1) = 1$ $$E_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \quad E_5 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$ $\dim(E_5) = 1$

so the geometric multiplicity of 1 is 1 and the geometric multiplicity of 5 is 1.

Rundown of Terminology in Examples

If $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$, then the characteristic polynomial is

$$\det(A - tI) = (t-1)(t-5)$$

The eigenvalues are 1 and 5. The arithmetic multiplicity of 1 is 1 and the arithmetic multiplicity of 5 is 1. The eigenspaces are

$$E_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \quad E_5 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

so the geometric multiplicity of 1 is 1 and the geometric multiplicity of 5 is 1. And we can see that A is diagonalizable for three reason

Rundown of Terminology in Examples

If $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$, then the characteristic polynomial is

$$\det(A - tI) = (t-1)(t-5)$$

The eigenvalues are 1 and 5. The arithmetic multiplicity of 1 is 1 and the arithmetic multiplicity of 5 is 1. The eigenspaces are

$$E_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \quad E_5 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

so the geometric multiplicity of 1 is 1 and the geometric multiplicity of 5 is 1. And we can see that A is diagonalizable for three reason

1. It has a set of 2 linearly independent eigenvectors

Rundown of Terminology in Examples

If $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$, then the characteristic polynomial is

$$\det(A - tI) = (t-1)(t-5)$$

The eigenvalues are 1 and 5. The arithmetic multiplicity of 1 is 1 and the arithmetic multiplicity of 5 is 1. The eigenspaces are

$$E_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \quad E_5 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

so the geometric multiplicity of 1 is 1 and the geometric multiplicity of 5 is 1. And we can see that A is diagonalizable for three reason

1. It has a set of 2 linearly independent eigenvectors
2. It has 2 distinct eigenvalues

Rundown of Terminology in Examples

If $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$, then the characteristic polynomial is

$$\det(A - tI) = (t-1)(t-5)$$

The eigenvalues are 1 and 5. The arithmetic multiplicity of 1 is 1 and the arithmetic multiplicity of 5 is 1. The eigenspaces are

$$E_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \quad E_5 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

so the geometric multiplicity of 1 is 1 and the geometric multiplicity of 5 is 1. And we can see that A is diagonalizable for three reason

1. It has a set of 2 linearly independent eigenvectors
2. It has 2 distinct eigenvalues
3. All geometric multiplicities are equal to the arithmetic multiplicities.

Rundown of Terminology in Examples

If $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

Rundown of Terminology in Examples

If $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, then the characteristic polynomial is

$$\det(A - tI) = (t-1)^2$$

Rundown of Terminology in Examples

If $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, then the characteristic polynomial is

$$\det(A - tI) = (t - 1)^2$$

A has only one eigenvalue, 1

Rundown of Terminology in Examples

If $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, then the characteristic polynomial is

$$\det(A - tI) = (t-1)^2$$

A has only one eigenvalue, 1, and it's arithmetic multiplicity if 2.

Rundown of Terminology in Examples

If $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, then the characteristic polynomial is

$$\det(A - tI) = (t-1)^2$$

A has only one eigenvalue, 1, and it's arithmetic multiplicity if 2. The eigenspaces is

$$E_1 = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

Rundown of Terminology in Examples

If $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, then the characteristic polynomial is

$$\det(A - tI) = (t-1)^2$$

A has only one eigenvalue, 1, and it's arithmetic multiplicity if 2. The eigenspaces is

dim $(E_1) = 1$

$$E_1 = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

so the geometric multiplicity of 1 is 1.

Rundown of Terminology in Examples

If $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, then the characteristic polynomial is

$$\det(A - tI) = (t-1)^2$$

A has only one eigenvalue, 1, and it's arithmetic multiplicity if 2. The eigenspaces is

$$E_1 = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

so the geometric multiplicity of 1 is 1.And we can see that A is not diagonalizable for two reasons

Rundown of Terminology in Examples

If $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, then the characteristic polynomial is

$$\det(A - tI) = (t-1)^2$$

A has only one eigenvalue, 1, and it's arithmetic multiplicity if 2. The eigenspaces is

$$E_1 = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

so the geometric multiplicity of 1 is 1.And we can see that A is not diagonalizable for two reasons

1. It only has a set of 1 linearly independent eigenvectors

Rundown of Terminology in Examples

If $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, then the characteristic polynomial is

$$\det(A - tI) = (t-1)^2$$

A has only one eigenvalue, 1, and it's arithmetic multiplicity if 2. The eigenspaces is

$$E_1 = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

so the geometric multiplicity of 1 is 1.And we can see that A is not diagonalizable for two reasons

1. It only has a set of 1 linearly independent eigenvectors
2. There is an eigenvalue whose geometric multiplicity is not the same as it's arithmetic multiplicity.

$a_1 = 2$ but $g_1 = 1$

so not diagonalizable

Rundown of Terminology in Examples

If

$$A = \begin{pmatrix} 1/2 & -1 & 1/2 \\ 0 & 1 & 0 \\ -3/2 & -3 & 5/2 \end{pmatrix}$$

Rundown of Terminology in Examples

If

$$A = \begin{pmatrix} 1/2 & -1 & 1/2 \\ 0 & 1 & 0 \\ -3/2 & -3 & 5/2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 2 \end{pmatrix}^{-1}$$

If

$$A = \begin{pmatrix} 1/2 & -1 & 1/2 \\ 0 & 1 & 0 \\ -3/2 & -3 & 5/2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 2 \end{pmatrix}^{-1}$$

then the characteristic polynomial is $\det(A - tI_3) = (t-1)^2(t-2)$

eigenvalue of 1 appearing twice

eigenvalue of 2 appearing once

Rundown of Terminology in Examples

If

$$A = \begin{pmatrix} 1/2 & -1 & 1/2 \\ 0 & 1 & 0 \\ -3/2 & -3 & 5/2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 2 \end{pmatrix}^{-1}$$

then the characteristic polynomial is $\det(A - tI_3) = (t-1)^2(t-2)$ and so we see that the eigenvalues are 1 and 2.

Rundown of Terminology in Examples

If

$$A = \begin{pmatrix} 1/2 & -1 & 1/2 \\ 0 & 1 & 0 \\ -3/2 & -3 & 5/2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 2 \end{pmatrix}^{-1}$$

then the characteristic polynomial is $\det(A - tI_3) = (t-1)^2(t-2)$ and so we see that the eigenvalues are 1 and 2. The arithmetic multiplicity of 1 is 2 and the arithmetic multiplicity of 2 is 1.

Rundown of Terminology in Examples

If

$$A = \begin{pmatrix} 1/2 & -1 & 1/2 \\ 0 & 1 & 0 \\ -3/2 & -3 & 5/2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 2 \end{pmatrix}^{-1}$$

then the characteristic polynomial is $\det(A - tI_3) = (t-1)^2(t-2)$ and so we see that the eigenvalues are 1 and 2. The arithmetic multiplicity of 1 is 2 and the arithmetic multiplicity of 2 is 1. The eigenspaces are $E_1 = \text{span}\{(1,0,1),(0,1,2)\}$, $E_2 = \text{span}\{(1,0,3)\}$.

Rundown of Terminology in Examples

If

$$A = \begin{pmatrix} 1/2 & -1 & 1/2 \\ 0 & 1 & 0 \\ -3/2 & -3 & 5/2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 2 \end{pmatrix}^{-1}$$

then the characteristic polynomial is $\det(A - tI_3) = (t-1)^2(t-2)$ and so we see that the eigenvalues are 1 and 2. The arithmetic multiplicity of 1 is 2 and the arithmetic multiplicity of 2 is 1. The eigenspaces are $E_1 = \text{span}\{(1,0,1),(0,1,2)\}$, $E_2 = \text{span}\{(1,0,3)\}$. The geometric multiplicity of 1 is 2 and the geometric multiplicity of 1 is 1.

$\dim(E_1) = 2$ $\qquad\qquad$ $\dim(E_2) = 1$

If

$$A = \begin{pmatrix} 1/2 & -1 & 1/2 \\ 0 & 1 & 0 \\ -3/2 & -3 & 5/2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 2 \end{pmatrix}^{-1}$$

then the characteristic polynomial is $\det(A - tI_3) = (t-1)^2(t-2)$ and so we see that the eigenvalues are 1 and 2. The arithmetic multiplicity of 1 is 2 and the arithmetic multiplicity of 2 is 1. The eigenspaces are $E_1 = \text{span}\{(1,0,1),(0,1,2)\}$, $E_2 = \text{span}\{(1,0,3)\}$. The geometric multiplicity of 1 is 2 and the geometric multiplicity of 1 is 1. And, we can see that A is diagonalizable for three reason:

Rundown of Terminology in Examples

If

$$A = \begin{pmatrix} 1/2 & -1 & 1/2 \\ 0 & 1 & 0 \\ -3/2 & -3 & 5/2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 2 \end{pmatrix}^{-1}$$

then the characteristic polynomial is $\det(A - tI_3) = (t-1)^2(t-2)$ and so we see that the eigenvalues are 1 and 2. The arithmetic multiplicity of 1 is 2 and the arithmetic multiplicity of 2 is 1. The eigenspaces are $E_1 = \text{span}\{(1,0,1),(0,1,2)\}$, $E_2 = \text{span}\{(1,0,3)\}$. The geometric multiplicity of 1 is 2 and the geometric multiplicity of 1 is 1. And, we can see that A is diagonalizable for three reason:

1. It has a set of 3 linearly independent eigenvectors

Rundown of Terminology in Examples

If

$$A = \begin{pmatrix} 1/2 & -1 & 1/2 \\ 0 & 1 & 0 \\ -3/2 & -3 & 5/2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 2 \end{pmatrix}^{-1}$$

then the characteristic polynomial is $\det(A - tI_3) = (t-1)^2(t-2)$ and so we see that the eigenvalues are 1 and 2. The arithmetic multiplicity of 1 is 2 and the arithmetic multiplicity of 2 is 1. The eigenspaces are $E_1 = \text{span}\{(1,0,1),(0,1,2)\}$, $E_2 = \text{span}\{(1,0,3)\}$. The geometric multiplicity of 1 is 2 and the geometric multiplicity of 1 is 1. And, we can see that A is diagonalizable for three reason:

1. It has a set of 3 linearly independent eigenvectors
2. All geometric multiplicities are equal to the arithmetic multiplicities.

$a_1 = 2$, $g_1 = 2$ $a_2 = 1$ $g_2 = 1$

Rundown of Terminology in Examples

If

$$A = \begin{pmatrix} 1/2 & -1 & 1/2 \\ 0 & 1 & 0 \\ -3/2 & -3 & 5/2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 2 \end{pmatrix}^{-1}$$

then the characteristic polynomial is $\det(A - tI_3) = (t-1)^2(t-2)$ and so we see that the eigenvalues are 1 and 2. The arithmetic multiplicity of 1 is 2 and the arithmetic multiplicity of 2 is 1. The eigenspaces are $E_1 = \text{span}\{(1,0,1),(0,1,2)\}$, $E_2 = \text{span}\{(1,0,3)\}$. The geometric multiplicity of 1 is 2 and the geometric multiplicity of 1 is 1. And, we can see that A is diagonalizable for three reason:

1. It has a set of 3 linearly independent eigenvectors
2. All geometric multiplicities are equal to the arithmetic multiplicities.
3. We were already given it in the form PDP^{-1}