SF 1684 Algebra and Geometry
Lecture 16

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Topics for Today

1. Linear Transformations in Different Bases
2. Change of Basis for Square Linear Transformations
3. Change of Basis for Non-Square Linear Transformations

Standard Matrix of a Linear Transformation

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$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad e_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th} \text{ position}$$

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NOTE: while T is a linear transformation $[T]$ is a matrix!

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$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} * \\ * \\ z \end{pmatrix}$$

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Can we describe this geometrically? If so, how?

Linear Transformation Not Under the Standard Basis

If $T : \mathbb{R}^n \to \mathbb{R}^n$, $[T]$ is called the standard matrix because we are using the standard basis $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ to define it.

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Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ a basis for $\mathbb{R}^n$ and let

$$A = \left([T(\vec{v}_1)]_B \quad [T(\vec{v}_2)]_B \quad \ldots \quad [T(\vec{v}_n)]_B \right)$$

Comment: if $B = \{e_1, e_2 \ldots e_n\}$ A standard matrix of T.

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$$[T(\vec{x})]_B = A[\vec{x}]_B$$

for every vector in $\vec{x} \in \mathbb{R}^n$.

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Then

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for every vector in $\vec{x} \in \mathbb{R}^n$. Moreover, A is the unique matrix with this property and we commonly denote $A = [T]_B$ and call it the **matrix of** T **with respect to the basis** B.

Proof

Wat to show $A := ([T(v_1)]_B \cdots [T(v_n)]_B)$, Then $[T(\vec{x})]_B = A[\vec{x}]_B$

$$[\vec{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \overset{\text{def}}{\Longrightarrow} \vec{x} = c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n$$

RHS: $$[T(\vec{x})]_B = [T(c_1\vec{v}_1 + \cdots + c_n \vec{v}_n)]_B = [c_1 T(\vec{v}_1) + \cdots + c_n T(\vec{v}_n)]_B$$

$$= c_1 [T(\vec{v}_1)]_B + \cdots + c_n [T(\vec{v}_n)]_B$$

LHS: $$A[\vec{x}]_B = ([T(v_1)]_B \cdots [T(v_n)]_B) \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = c_1 [T(v_1)]_B + \cdots + c_n [T(v_n)]_B$$

RHS = LHS

done.

Example

Exercise

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with *standard* matrix

$$[T] = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

Find $[T]_B$, the matrix of T with respect to the basis $B = \{\vec{v}_1, \vec{v}_2\}$ where

$$\vec{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Use it to describe T geometrically and calculate $\vec{e}_1$.

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$$[T]_B = \big([T(\vec{v}_1)]_B \quad [T(\vec{v}_2)]_B\big)$$

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Hence, we need to find the coordinates of $T(\vec{v}_1)$ and $T(\vec{v}_2)$ with respect to B.

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Hence we see that

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In particular, $[T]_B$ is diagonal!

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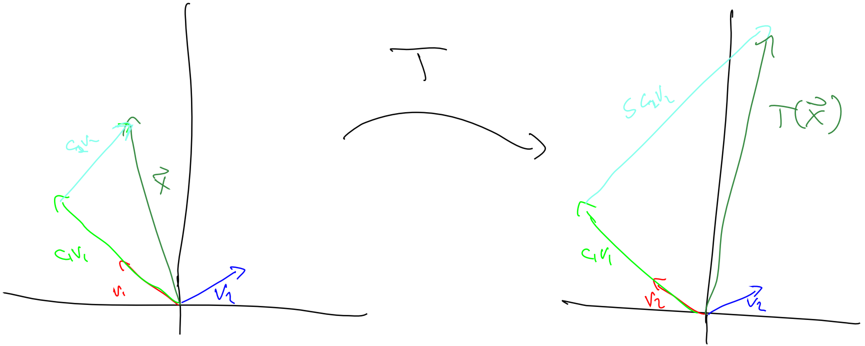
Likewise, $[T]_B$ being diagonal corresponds to stretching along the basis vectors B.

Example Continued

That is, we may conclude that T acts by stretching along the direction of $\vec{v}_1$ by a factor of 1 and stretching along the direction of $\vec{v}_2$ by a factor of 5.



Comment: This easy geometric interpretation works only because $[T]_B$ was diagonal!

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Now, to use this to calculate $T(\vec{e}_1)$, we need to write $\vec{e}_1$ in the basis B.

$$[T(e_1)]_B = [T]_B [e_1]_B$$

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$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

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That is, we may conclude that T acts by stretching along the direction of $\vec{v}_1$ by a factor of 1 and stretching along the direction of $\vec{v}_2$ by a factor of 5.

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$$[T(\vec{x})]_B = [T]_B[\vec{x}]_B = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 15 \end{bmatrix} \neq T(\vec{x})$$

$T(\vec{x}) = 2\vec{v}_1 + 15\vec{v}_2$

Change of Basis of Linear Transformation

Question

If $T : \mathbb{R}^n \to \mathbb{R}^n$ and B, B' are two bases for $\mathbb{R}^n$, how are $[T]_B$ and $[T]_{B'}$ related?

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Question

If $T : \mathbb{R}^n \to \mathbb{R}^n$ and B, B' are two bases for $\mathbb{R}^n$, how are $[T]_B$ and $[T]_{B'}$ related?

Theorem

If $T : \mathbb{R}^n \to \mathbb{R}^n$ and $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ and $B' = \{\vec{v}_1', \dots, \vec{v}_n'\}$ are two bases for $\mathbb{R}^n$, then

$$[T]_{B'} = P[T]_B P^{-1}$$

where

$$P = P_{B \to B'} = \begin{pmatrix} [\vec{v}_1]_{B'} & [\vec{v}_2]_{B'} & \dots & [\vec{v}_n]_{B'} \end{pmatrix}$$

is the transition matrix from $B \to B'$.

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can be thought of doing three things to a vector in base B':

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3. Changes the resulting vector back from base B to B' $\leftarrow P$

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3. Changes the resulting vector back from base B to B'

So it makes sense that this would be the same as just applying T in base B'.

Transition Between Orthonormal Bases

Corollary

If $T : \mathbb{R}^n \to \mathbb{R}^n$ *and* $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ *and* $B' = \{\vec{v}_1', \ldots, \vec{v}_n'\}$ *are two orthonormal bases for* $\mathbb{R}^n$*, then*

$$[T]_{B'} = P[T]_B P^T$$

where $P = P_{B \to B'}$ *is the transition matrix from* $B \to B'$.

Theorem says P^{-1}

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If B and B' are orthonormal bases then $P_{B \to B'}$ is an orthogonal matrix. (Exercise: show this)

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If B and B' are orthonormal bases then $P_{B \to B'}$ is an orthogonal matrix. (Exercise: show this)

Theorem:

Hence $P^T P = I_n$ and so $P^{-1} = P^T$.

$[T]_{B'} = P [T]_B P^{-1}$ □

orthon $\Rightarrow P^{-1} = P^T$

Transition to and from Standard Basis

Corollary

If $T : \mathbb{R}^n \to \mathbb{R}^n$ *and* $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ *is a basis for* $\mathbb{R}^n$ *and* S *is the standard basis, then*

$$[T]_S = [T] = P[T]_B P^{-1}$$

where

$S = \{e_1, \ldots e_n\}$

$$P = P_{B\to S} = \begin{pmatrix} [\vec{v}_1]_S & [\vec{v}_2]_S & \ldots & [\vec{v}_n]_S \end{pmatrix}$$

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Moreover, if B *is an orthonormal basis, then*

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Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation with standard matrix $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

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$S =$ standard basis $= \{e_1, e_2\}$

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And a quick calculation confirms that

$$\underbrace{\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}}_{[T]} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{P} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}}_{[T]_B} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{P^{-1}}$$

Issue with Non-Square Transformations

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Up until now, we have been only discussing transformations from $\mathbb{R}^n \to \mathbb{R}^n$. This was necessary as if we if $T : \mathbb{R}^n \to \mathbb{R}^m$, then $\vec{x} \in \mathbb{R}^n$ but $T(\vec{x}) \in \mathbb{R}^m$ and if we have a basis for $\mathbb{R}^n$, then

$$[T(\vec{x})]_B$$

because $T(\vec{x}) \in \mathbb{R}^m$

would make no sense.

If $T: \mathbb{R}^n \to \mathbb{R}^n$

$$[T(x)]_B = [T]_B [\vec{x}]_B$$

Issue with Non-Square Transformations

Up until now, we have been only discussing transformations from $\mathbb{R}^n \to \mathbb{R}^n$. This was necessary as if we if $T : \mathbb{R}^n \to \mathbb{R}^m$, then $\vec{x} \in \mathbb{R}^n$ but $T(\vec{x}) \in \mathbb{R}^m$ and if we have a basis for $\mathbb{R}^n$, then

$$[T(\vec{x})]_B$$

would make no sense. Whereas if we tried to use a basis B' of $\mathbb{R}^m$ so that $[T(\vec{x})]_{B'}$ makes sense, we would now have that

$$\underline{[\vec{x}]_{B'}} \qquad x \in \mathbb{R}^n \neq \mathbb{R}^m$$

makes no sense.

$$[T(\vec{x})]_B = [T]_B [\vec{x}]_B$$

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Up until now, we have been only discussing transformations from $\mathbb{R}^n \to \mathbb{R}^n$. This was necessary as if we if $T : \mathbb{R}^n \to \mathbb{R}^m$, then $\vec{x} \in \mathbb{R}^n$ but $T(\vec{x}) \in \mathbb{R}^m$ and if we have a basis for $\mathbb{R}^n$, then

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would make no sense. Whereas if we tried to use a basis B' of $\mathbb{R}^m$ so that $[T(\vec{x})]_{B'}$ makes sense, we would now have that

$$[\vec{x}]_{B'}$$

makes no sense.

Conclusion: using only one basis there is no way to make sense of the statement

$$[(T(\vec{x})]_B = [T]_B[\vec{x}]_B$$

if $T : \mathbb{R}^n \to \mathbb{R}^m$ for $n \neq m$.

Non-Square Linear Transformation With Respect to Two Bases

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ *and let* $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ *be a basis for* $\mathbb{R}^n$ *and* $B' = \{\vec{v}'_1, \ldots, \vec{v}'_m\}$ *be a basis for* $\mathbb{R}^m$

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$$A = \left([T(\vec{v}_1)]_{B'} \quad [T(\vec{v}_2)]_{B'} \quad \ldots \quad [T(\vec{v}_n)]_{B'} \right)$$

and get that

$$[T(\vec{x})]_{B'} = A[\vec{x}]_B$$

for every vector $\vec{x} \in \mathbb{R}^n$.

different bases

Non-Square Linear Transformation With Respect to Two Bases

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ *and let* $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ *be a basis for* $\mathbb{R}^n$ *and* $B' = \{\vec{v}_1', \ldots, \vec{v}_m'\}$ *be a basis for* $\mathbb{R}^m$ *then we define*

$$A = \begin{pmatrix} [T(\vec{v}_1)]_{B'} & [T(\vec{v}_2)]_{B'} & \ldots & [T(\vec{v}_n)]_{B'} \end{pmatrix}$$

and get that

$$[T(\vec{x})]_{B'} = A[\vec{x}]_B$$

for every vector $\vec{x} \in \mathbb{R}^n$*. We denote the matrix* $A = [T]_{B',B}$ *ad call it the* **matrix for** T **with respect to the bases** B **and** B'.

Non-Square Linear Transformation With Respect to Two Bases

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ *and let* $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ *be a basis for* $\mathbb{R}^n$ *and* $B' = \{\vec{v}'_1, \ldots, \vec{v}'_m\}$ *be a basis for* $\mathbb{R}^m$ *then we define*

$$A = \begin{pmatrix} [T(\vec{v}_1)]_{B'} & [T(\vec{v}_2)]_{B'} & \ldots & [T(\vec{v}_n)]_{B'} \end{pmatrix}$$

we don't need $n \neq m$

and get that

$$[T(\vec{x})]_{B'} = A[\vec{x}]_B$$

for every vector $\vec{x} \in \mathbb{R}^n$. *We denote the matrix* $A = [T]_{B',B}$ *ad call it the* **matrix for** T **with respect to the bases** B **and** B'.

Remark

If $T : \mathbb{R}^n \to \mathbb{R}^n$, and B is a basis for $\mathbb{R}^n$, then this new notation is consistent with our old notation in that $[T]_B = [T]_{B,B}$.

Example

Exercise

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation define by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix}$$

Let $B = \{\vec{v}_1, \vec{v}_2\}$ be a basis for $\mathbb{R}^2$ and $B' = \{\vec{v}_1', \vec{v}_2', \vec{v}_3'\}$ be a basis for $\mathbb{R}^3$ where

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \vec{v}_1' = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \vec{v}_2' = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \vec{v}_3' = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

Find $[T]_{B',B}$.

Solution

We know that

$$[T]_{B',B} = \left([T(\vec{v}_1)]_{B'} \quad [T(\vec{v}_2)]_{B'} \right)$$

$B = \{ v_1, v_2 \}$

$B' = \{ v_1', v_2', v_3' \}$

Solution

We know that

$$[T]_{B',B} = \big([T(\vec{v}_1)]_{B'} \quad [T(\vec{v}_2)]_{B'}\big)$$

and can calculate

$$T(\vec{v}_1) = T\left(\underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{\vec{v}_1}\right)$$

Solution

We know that

$$[T]_{B',B} = \left([T(\vec{v}_1)]_{B'} \quad [T(\vec{v}_2)]_{B'} \right)$$

and can calculate

$$T(\vec{v}_1) = T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ -5(3) + 13(1) \\ -7(3) + 16(1) \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix} \right)$$

Solution

We know that

$$[T]_{B',B} = \left([T(\vec{v}_1)]_{B'} \quad [T(\vec{v}_2)]_{B'} \right)$$

and can calculate

$$T(\vec{v}_1) = T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -5(3) + 13(1) \\ -7(3) + 16(1) \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} = c_1 v_1' + c_2 v_2' + c_3 v_3'$$

write in basis B'

Solution

We know that

$$[T]_{B',B} = \big([T(\vec{v}_1)]_{B'} \quad [T(\vec{v}_2)]_{B'}\big)$$

and can calculate

$$T(\vec{v}_1) = T\left(\begin{bmatrix}3\\1\end{bmatrix}\right) = \begin{bmatrix}1\\-5(3)+13(1)\\-7(3)+16(1)\end{bmatrix} = \begin{bmatrix}1\\-2\\-5\end{bmatrix} = -\underset{\substack{\uparrow\\ v_2}}{\begin{bmatrix}-1\\2\\2\end{bmatrix}} - \frac{5}{2}\underset{\substack{\uparrow\\ v_3}}{\begin{bmatrix}0\\0\\2\end{bmatrix}}$$

$$= -\vec{v}_2' - \frac{5}{2}\vec{v}_3'$$

Solution

We know that

$$[T]_{B',B} = \left([T(\vec{v}_1)]_{B'} \quad [T(\vec{v}_2)]_{B'}\right)$$

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$0\vec{v}_1'$

$$= -\vec{v}_2' - \frac{5}{2}\vec{v}_3' \implies [T(\vec{v}_1)]_{B'} = \begin{bmatrix} 0 \\ -1 \\ -\frac{5}{2} \end{bmatrix}$$

Solution

We know that

$$[T]_{B',B} = \left([T(\vec{v}_1)]_{B'} \quad [T(\vec{v}_2)]_{B'}\right)$$

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$$= -\vec{v}_2' - \frac{5}{2}\vec{v}_3' \implies [T(\vec{v}_1)]_{B'} = \begin{bmatrix}0\\-1\\-\frac{5}{2}\end{bmatrix}$$

$$T(\vec{v}_2) = T\left(\begin{bmatrix}5\\2\end{bmatrix}\right) = \begin{bmatrix}2\\1\\-3\end{bmatrix}$$

v_2

Solution

We know that

$$[T]_{B',B} = \begin{pmatrix}[T(\vec{v}_1)]_{B'} & [T(\vec{v}_2)]_{B'}\end{pmatrix}$$

and can calculate

$$T(\vec{v}_1) = T\left(\begin{bmatrix}3\\1\end{bmatrix}\right) = \begin{bmatrix}1\\-5(3)+13(1)\\-7(3)+16(1)\end{bmatrix} = \begin{bmatrix}1\\-2\\-5\end{bmatrix} = -\begin{bmatrix}-1\\2\\2\end{bmatrix} - \frac{5}{2}\begin{bmatrix}0\\0\\2\end{bmatrix}$$

$$= -\vec{v}_2' - \frac{5}{2}\vec{v}_3' \implies [T(\vec{v}_1)]_{B'} = \begin{bmatrix}0\\-1\\-\frac{5}{2}\end{bmatrix}$$

$$T(\vec{v}_2) = T\left(\begin{bmatrix}5\\2\end{bmatrix}\right) = \begin{bmatrix}2\\1\\-3\end{bmatrix} = \frac{5}{2}\begin{bmatrix}1\\0\\-1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}-1\\2\\2\end{bmatrix} - \frac{3}{4}\begin{bmatrix}0\\0\\2\end{bmatrix}$$

v_1' v_2' v_3'

$$= \frac{5}{2}\vec{v}_1' + \frac{1}{2}\vec{v}_2' - \frac{3}{4}\vec{v}_3'$$

Solution

We know that

$$[T]_{B',B} = \left([T(\vec{v}_1)]_{B'} \quad [T(\vec{v}_2)]_{B'} \right)$$

and can calculate

$$T(\vec{v}_1) = T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -5(3) + 13(1) \\ -7(3) + 16(1) \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} = -\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} - \frac{5}{2}\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$= -\vec{v}_2' - \frac{5}{2}\vec{v}_3' \implies [T(\vec{v}_1)]_{B'} = \begin{bmatrix} 0 \\ -1 \\ -\frac{5}{2} \end{bmatrix}$$

$$T(\vec{v}_2) = T\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \frac{5}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} - \frac{3}{4}\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$= \frac{5}{2}\vec{v}_1' + \frac{1}{2}\vec{v}_2' - \frac{3}{4}\vec{v}_3' \implies [T(\vec{v}_{\cancel{1}\,2})]_{B'} = \begin{bmatrix} \frac{5}{2} \\ \frac{1}{2} \\ -\frac{3}{4} \end{bmatrix}$$

Solution 2

Thus we conclude that

$$[T]_{B',B} = \left([T(\vec{v}_1)]_{B'} \quad [T(\vec{v}_2)]_{B'} \right)$$

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$[T(\vec{v}_1)]_{B'}$ $[T(\vec{v}_0)]_{B'}$

Solution 2

Thus we conclude that

$$[T]_{B',B} = \begin{pmatrix} [T(\vec{v}_1)]_{B'} & [T(\vec{v}_2)]_{B'} \end{pmatrix} = \begin{pmatrix} 0 & \frac{5}{2} \\ -1 & \frac{1}{2} \\ -\frac{3}{2} & -\frac{3}{4} \end{pmatrix}$$

Therefore, since

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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Therefore, since

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{v_1} - \underbrace{\begin{bmatrix} 5 \\ 2 \end{bmatrix}}_{v_2} = 2\vec{v}_1 - \vec{v}_2$$

Solution 2

Thus we conclude that

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and so

$$[T(\vec{e}_1)]_{B'}$$

Solution 2

Thus we conclude that

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and so

$$[T(\vec{e}_1)]_{B'} = [T]_{B',B}[\vec{e}_1]_B$$

Solution 2

Thus we conclude that

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and so

$$[T(\vec{e}_1)]_{B'} = [T]_{B',B}[\vec{e}_1]_B = \begin{pmatrix} 0 & \frac{5}{2} \\ -1 & \frac{1}{2} \\ -\frac{3}{2} & -\frac{3}{4} \end{pmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

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and so

$T(\vec{e}_1)$

$$* \quad [T(\vec{e}_1)]_{B'} = [T]_{B',B}[\vec{e}_1]_B = \begin{pmatrix} 0 & \frac{5}{2} \\ -1 & \frac{1}{2} \\ -\frac{3}{2} & -\frac{3}{4} \end{pmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \underline{\begin{bmatrix} -\frac{5}{2} \\ -\frac{5}{2} \\ -\frac{9}{4} \end{bmatrix}}$$

$$T(\vec{e}_1) = -\frac{5}{2} v_1' - \frac{5}{2} v_2' - \frac{9}{4} v_3'$$

check: that * works with the initial defn of T.

Changing Two Bases for Non-Square Linear Transformations

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ *be a linear transformation. Let* B_1 *and* B_2 *be bases for* $\mathbb{R}^n$ *and* B_1', B_2' *be bases for* $\mathbb{R}^m$*. Then*

$$[T]_{B_1',B_1} = P_{B_2' \to B_1'} [T]_{B_2',B_2} P^{-1}_{B_2 \to B_1}$$

Changing Two Bases for Non-Square Linear Transformations

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let B_1 and B_2 be bases for $\mathbb{R}^n$ and B_1', B_2' be bases for $\mathbb{R}^m$. Then

$$[T]_{B_1',B_1} = P_{B_2' \to B_1'} [T]_{B_2',B_2} P^{-1}_{B_2 \to B_1}$$

not the same matrix!!!

Since $P^{-1}_{B_2 \to B_1} = P_{B_1 \to B_2}$, the right hand side can be thought of as three different operations:

Changing Two Bases for Non-Square Linear Transformations

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Since $P^{-1}_{B_2\to B_1} = P_{B_1\to B_2}$, the right hand side can be thought of as three different operations:

1. Changing the $\mathbb{R}^n$ basis from B_1 to B_2 $\leftarrow$ $P^{-1}_{B_2\to B_1}$

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Since $P^{-1}_{B_2 \to B_1} = P_{B_1 \to B_2}$, the right hand side can be thought of as three different operations:

1. Changing the $\mathbb{R}^n$ basis from B_1 to B_2 $\leftarrow$ $P^{-1}_{B_2 \to B_1}$
2. Applying T from basis B_2 into basis B_2' $\leftarrow$ $[T]_{B_2', B_2}$

Changing Two Bases for Non-Square Linear Transformations

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Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let B_1 and B_2 be bases for $\mathbb{R}^n$ and B_1', B_2' be bases for $\mathbb{R}^m$. Then

$$[T]_{B_1', B_1} = \underline{P_{B_2' \to B_1'} [T]_{B_2', B_2} P^{-1}_{B_2 \to B_1}}$$

Since $P^{-1}_{B_2 \to B_1} = P_{B_1 \to B_2}$, the right hand side can be thought of as three different operations:

1. Changing the $\mathbb{R}^n$ basis from B_1 to B_2 $\longleftarrow P^{-1}_{B_2 \to B_1}$
2. Applying T from basis B_2 into basis B_2' $\longleftarrow [T]_{B_2', B_2}$
3. Changing the $\mathbb{R}^m$ from basis B_2' to B_1' $\longleftarrow P_{B_2' \to B_1'}$

Changing Two Bases for Non-Square Linear Transformations

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let B_1 and B_2 be bases for $\mathbb{R}^n$ and B_1', B_2' be bases for $\mathbb{R}^m$. Then

$$[T]_{B_1',B_1} = P_{B_2' \to B_1'}[T]_{B_2',B_2} P^{-1}_{B_2 \to B_1}$$

Since $P^{-1}_{B_2 \to B_1} = P_{B_1 \to B_2}$, the right hand side can be thought of as three different operations:

1. Changing the $\mathbb{R}^n$ basis from B_1 to B_2
2. Applying T from basis B_2 into basis B_2'
3. Changing the $\mathbb{R}^m$ from basis B_2' to B_1'

Hence, it makes sense this should be applying T from basis B_1 to B_1'.