SF 1684 Algebra and Geometry
Lecture 15

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Topics for Today

1. Least Squares Process
2. Change of Basis
3. Gram-Schmidt Process

Approximate Solutions to Matrix Equations

For a given $m \times n$ matrix A, and a vector $\vec{b}$ in $\mathbb{R}^m$ we are interested in finding solutions to $\vec{x}$ in $\mathbb{R}^n$ such that $A\vec{x} = \vec{b}$.

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If A is an $m \times n$ matrix and $\vec{b}$ is a vector in $\mathbb{R}^m$, then a vector $\vec{x}$ in $\mathbb{R}^n$ is called a **best approximate solution** or a **least squares solution** to $A\vec{x} = \vec{b}$ if

$$\underbrace{\left\| \vec{b} - A\vec{x} \right\|}_{\text{distance between } A\vec{x} \text{ \& } \vec{b}} \leq \underbrace{\left\| \vec{b} - A\vec{y} \right\|}_{\text{distance between } A\vec{y} \text{ \& } \vec{b}}$$

for all $\vec{y}$ in $\mathbb{R}^n$

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for all $\vec{y}$ in $\mathbb{R}^n$. The vector $\vec{b} - A\vec{x}$ is called the **least squares error vector**

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for all $\vec{y}$ in $\mathbb{R}^n$. The vector $\vec{b} - A\vec{x}$ is called the **least squares error vector**, and the scalar $\left\|\vec{b} - A\vec{x}\right\|$ is called the **least squares error**.

Remark: the least square error is 0 iff there is a solution to $Ax = b$.

Why "least squares"?

Note that if we write

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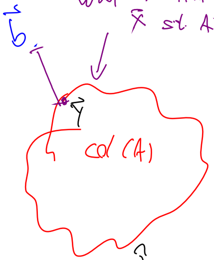
Note that the set $\{A\vec{x} : \vec{x} \in \mathbb{R}^n\}$ is the column space of A.

Why "least squares"?

If $A = [c_1 \dots c_n]$ then $\mathrm{col}(A) = \mathrm{Span}(c_1, \dots, c_n)$

want to find this $\vec{x}$ s.t. $A\vec{x} = \vec{y}$



all the $\vec{y}$ that have a solution to $A\vec{x} = \vec{y}$

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then we get

$$\left\| \vec{b} - A\vec{x} \right\| = \sqrt{c_1^2 + c_2^2 + \cdots + c_m^2}$$

And we are wishing to *minimize* this value.

Note that the set $\{A\vec{x} : \vec{x} \in \mathbb{R}^n\}$ is the column space of A. Hence, we are really looking for the "minimal distance between the vector $\cancel{\vec{x}}$ $\vec{b}$ and the subspace col(A)"

Distance to a Subspace

Question (Minimal Distance to a Subspace)

Given a subspace W of $\mathbb{R}^n$ and a vector $\vec{b} \in \mathbb{R}^n$, can we find a vector $\vec{w}$ in W that is closest to $\vec{b}$ in the sense that

$$\left\|\vec{w} - \vec{b}\right\| \leq \left\|\vec{v} - \vec{b}\right\|$$

distance between $\vec{w}$ & $\vec{b}$

distance between $\vec{v}$ & $\vec{b}$

for all $\vec{v}$ in W?

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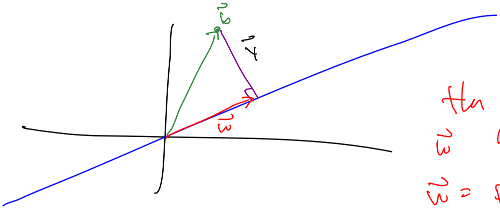
for all $\vec{v}$ in W? Such a vector $\vec{w}$ is called a **best approximation to $\vec{b}$ from W**.

$n=2$ $W =$ space of $\mathbb{R}^2$ $\rightarrow$ $W =$ line $= \text{span}\{\vec{v}\}$



$\|\vec{y}\|$ is the shortest distance from $\vec{b}$ to line

then $\|\vec{w} - \vec{b}\| = \|\vec{y}\|$ shortest

$\vec{w}$ best app to $\vec{b}$

$\vec{w} = \text{proj}_{\text{Line}}\, \vec{b}$

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Theorem (Best Approximation Theorem)

If W is a subspace of $\mathbb{R}^n$ and $\vec{b}$ is a vector in $\mathbb{R}^n$, then there is a unique best approximation to $\vec{b}$ from W

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Theorem (Best Approximation Theorem)

If W is a subspace of $\mathbb{R}^n$ and $\vec{b}$ is a vector in $\mathbb{R}^n$, then there is a unique best approximation to $\vec{b}$ from W, namely $\vec{w} = proj_W \vec{b}$.

Solving Least Squares

So, given an $m \times n$ matrix A and a vector $\vec{b}$ in $\mathbb{R}^m$, we want to find a vector $\vec{x}$ in $\mathbb{R}^n$ such that $A\vec{x}$ is "closest" to $\vec{b}$.

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It is NOT the case that $\text{proj}_{\text{col}(A)}\vec{b} = A(A^T A)^{-1}A^T\vec{b}$.

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Theorem

The least squares solutions of a linear system $A\vec{x} = \vec{b}$ are the exact solutions to the equation

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We have seen that it is enough to solve $A\vec{x} = \mathrm{proj}_{\mathrm{col}(A)}\vec{b}$. Moreover, we know that we can write

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$$A^T A\vec{x} = A^T\text{proj}_{\text{col}(A)}\vec{b} = A^T(\vec{b} - \text{proj}_{\text{null}(A^T)}\vec{b})$$

$= A^T\vec{b} - A^T(\text{proj}_{\text{null}\,A^T} b)$

$= A^T\vec{b} - 0$

$= A^T\vec{b}$

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Example

Find the least squares solution and least squares error for the linear system

$$3x_1 + 2x_2 - x_3 = 2$$

$$x_1 - 4x_2 + 3x_3 = -2$$

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$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & -4 & 3 \\ 1 & 10 & -7 \end{pmatrix} \quad \vec{b} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \quad A^T = \begin{pmatrix} 3 & 1 & 1 \\ 2 & -4 & 10 \\ -1 & 3 & -7 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 11 & 12 & -7 \\ 12 & 120 & -84 \\ -7 & -84 & 59 \end{pmatrix}$$

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$$A^TA = \begin{pmatrix} 11 & 12 & -7 \\ 12 & 120 & -84 \\ -7 & -84 & 59 \end{pmatrix} \qquad A^T\vec{b} = \begin{bmatrix} 5 \\ 22 \\ -15 \end{bmatrix}$$

Example continued

Hence we need to solve $A^T A\tilde{\vec{x}} = A^T\vec{b}$ and so putting it in an augmented matrix we get

$$(A^T A | A^T\vec{b})$$

Example continued

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$$(A^T A | A^T \vec{b}) = \left(\begin{array}{ccc|c} 11 & 12 & -7 & 5 \\ 12 & 120 & -84 & 22 \\ -7 & -84 & 59 & -15 \end{array}\right)$$

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And we see that

$$\vec{x} = \begin{bmatrix} 2/7 - t/7 \\ 13/84 + 5t/7 \\ t \end{bmatrix}$$

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$$\vec{x} = \begin{bmatrix} 2/7 - t/7 \\ 13/84 + 5t/7 \\ t \end{bmatrix} = \begin{bmatrix} 2/7 \\ 13/84 \\ 0 \end{bmatrix} + \begin{bmatrix} 1/7 \\ 5/7 \\ 1 \end{bmatrix} t$$

is a least squares solution for any t.

Check whether $A\begin{bmatrix} 2/7 \\ 13/84 \\ 0 \end{bmatrix} = \vec{b}$

If so have a solution. If not have an app solution.

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To find the least squares error, it is enough now to find $\left\|\vec{b} - A\vec{x}\right\|$ for *any* of the $\vec{x}$ we found above. (Exercise: Why does it not matter which $\vec{x}$ we take?)

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$$\left\|\vec{b} - A\vec{x}\right\| = \left\| \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} - \begin{pmatrix} 3 & 2 & -1 \\ 1 & -4 & 3 \\ 1 & 10 & -7 \end{pmatrix} \begin{bmatrix} 2/7 \\ 13/84 \\ 0 \end{bmatrix} \right\|$$

Example Continued

To find the least squares error, it is enough now to find $\left\|\vec{b} - A\vec{x}\right\|$ for *any* of the $\vec{x}$ we found above. (Exercise: Why does it not matter which $\vec{x}$ we take?) So, setting $t = 0$, we get

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Remark

The least squares error of a linear system will be 0 if and only if there is a solution to $A\vec{x} = \vec{b}$

Coordinates of Vectors in Other Basis

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $e_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

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So, what to do with these new numbers $2, 2, -1$?

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$$(\vec{w})_B = (a_1, a_2, \ldots, a_k) \qquad \text{or} \qquad [\vec{w}]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}$$

$$[w]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} \Longleftrightarrow W = a_1\vec{v}_1 + \cdots + a_k\vec{v}_k$$

Example

If we define the two bases we had above as

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad \text{and} \qquad B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

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Then we can write our vector $\vec{v} = (1, 0, 3)$ as

$$[\vec{v}]_S = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \text{ since } \vec{v} = 1 \underset{e_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} + 0 \underset{e_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} + 3 \underset{e_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

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but

$$[\vec{v}]_B = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \text{ since } \vec{v} = 2 \underset{v_1}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} + 2 \underset{v_2}{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}} + (-1) \underset{v_3}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$$

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because $1 \cdot \begin{bmatrix}1\\0\\1\end{bmatrix} + 0 \begin{pmatrix}0\\1\\2\end{pmatrix} + 0 \cdot \begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix}$

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Remarks

If $B = \{\vec{v}_1, \ldots, \vec{v}_k\}$, then

$$[\vec{w}]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \iff \vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k$$

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If $B = \{\vec{v}_1, \dots, \vec{v}_k\}$, then

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Hence, for any of the $\vec{v}_i$, we get

i^{th} coordinate

$$[\vec{v}_i]_B = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \vec{e}_i \text{ since } \vec{v}_i = 0\vec{v}_1 + 0\vec{v}_2 + \cdots + 1\vec{v}_i + \cdots + 0\vec{v}_k$$

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That is, looking at vectors with respect to a certain basis can simplify matters.

Change of Basis Problem

Question

If $\vec{w}$ is a vector in $\mathbb{R}^n$, and if we change the basis for $\mathbb{R}^n$ from a basis B to a basis B', how are $[\vec{w}]_B$ and $[\vec{w}]_{B'}$ related?

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Now, let $\vec{w}$ be any vector, then we have

$$[\vec{w}]_B = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \overset{\text{def}}{\iff} \vec{w} = k_1\vec{v}_1 + k_2\vec{v}_2$$

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Change of Basis Problem 2

Expanding and collecting like terms we see that

$$[\vec{w}]_B = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \iff \vec{w} = (ak_1 + ck_2)\vec{v}_1' + (bk_1 + dk_2)\vec{v}_2'$$

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$$\underbrace{\begin{bmatrix} ak_1 + ck_2 \\ bk_1 + dk_2 \end{bmatrix}}_{[w]_{B'}} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \underbrace{\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}}_{(w)_B}$$

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And we see that the columns of our matrices are exactly $[\vec{v}_1]_{B'}$ and $[\vec{v}_2]_{B'}$

$[v_1]_{B'}$ $[v_2]_{B'}$

Change of Basis Theorem

Theorem (Change of Basis Theorem)

If $\vec{w}$ is a vector in $\mathbb{R}^n$ and if $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ and $B' = \{\vec{v}'_1, \ldots, \vec{v}'_n\}$ are bases for $\mathbb{R}^n$, then

$$[\vec{w}]_{B'} = P_{B\to B'}[\vec{w}]_B$$

where $P_{B\to B'}$ is a matrix whose columns are the vectors of B in the bassi B':

$$P_{B\to B'} = \begin{pmatrix} [\vec{v}_1]_{B'} & [\vec{v}_2]_{B'} & \ldots & [\vec{v}_n]_{B'} \end{pmatrix}$$

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The matrix $P_{B\to B'}$ is called the **transition matrix** *(or the* **change of coordinates matrix***) from B to B'.*

Example

Find the change of coordinate matrix $P_{B \to B'}$ and $P_{B' \to B}$

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \text{ to } B' = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

Example

Find the change of coordinate matrix $P_{B\to B'}$ and $P_{B'\to B}$

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \text{ to } B' = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

Now, we know that

$$P_{B\to B'} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{B'} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{B'} \right)$$

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Further, we see that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

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$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{B'} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Example

Find the change of coordinate matrix $P_{B\to B'}$ and $P_{B'\to B}$

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \text{ to } B' = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

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$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

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Find the change of coordinate matrix $P_{B \to B'}$ and $P_{B' \to B}$

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Now, we know that

$$P_{B \to B'} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{B'} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{B'} \right) = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

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Example continued

And so, we conclude that

$$P_{B \to B'} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

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$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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Example continued

And so, we conclude that

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Same here is not surprising since $B = \{e_1, e_2\}$ the standard basis.

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and so

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and so

$$P_{B\to B'} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

Invertible Change of Basis

Theorem

If B and B' are two basis, then the change of basis matrices $P_{B \to B'}$ and $P_{B' \to B}$ are invertible and each other's inverse.

Invertible Change of Basis

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If B and B' are two basis, then the change of basis matrices $P_{B\to B'}$ and $P_{B'\to B}$ are invertible and each other's inverse. That is:

$$P^{-1}_{B\to B'} = P_{B'\to B} \text{ and } P^{-1}_{B'\to B} = P_{B\to B'}$$

$P_{B\to B'}$ goes from B to B'

$P_{B'\to B}$ goes from B' to B

So $P_{B'\to B} \cdot P_{B\to B'}$ goes from B to B' to B

Hence $P_{B'\to B} \cdot P_{B\to B'}$ is the transition from $B \to B$ which is identity.

Invertible Change of Basis

Theorem

If B and B' are two basis, then the change of basis matrices $P_{B\to B'}$ and $P_{B'\to B}$ are invertible and each other's inverse. That is:

$$P_{B\to B'}^{-1} = P_{B'\to B} \text{ and } P_{B'\to B}^{-1} = P_{B\to B'}$$

Exercise

Show that the two matrices we found from the previous example

$$\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

are inverses of each other.

Algorithm for Computing $P_{B \to B'}$

Let B and B' be two bases.

Algorithm for Computing $P_{B \to B'}$

$B = (v_1 \dots v_k)$ $B' = \{v_1', \dots v_k'\}$

$(B|B') = \left(v_1 \dots v_k \,\middle|\, v_1' \dots v_k' \right)$

Let B and B' be two bases.

1. Form the matrix $(B|B')$ where the columns of B are the vectors in basis B and the columns of B' are the vectors in B'

Algorithm for Computing $P_{B \to B'}$

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Algorithm for Computing $P_{B \to B'}$

Let B and B' be two bases.

1. Form the matrix $(B|B')$ where the columns of B are the vectors in basis B and the columns of B' are the vectors in B'
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3. The resulting matrix will be $(I|P_{B \to B'})$

Orthogonal and Orthonormal Basis

As we have seen, working with some basis gives us an advantage.

Orthogonal and Orthonormal Basis

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Definition

We say a basis $B = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is **orthogonal** if

$$\vec{v}_i \cdot \vec{v}_j = 0 \text{ for all } i \neq j.$$

all vectors in the basis are orthogonal to each other (perpindicular)

$V_1 \cdot V_2 = 0$

$V_1 \cdot V_3 = 0$

$V_3 \cdot V_5 = 0$

$\vdots$

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We say the basis is **orthonormal** if it is orthogonal plus

$$\|v_i\| \qquad \|\vec{v}_i\| = 1 \text{ for all } i.$$

all vectors in the basis are normal (unit length).

Orthogonal and Orthonormal Basis

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$$\|\vec{v}_i, \vec{v}_i\| = 1 \text{ for all } i.$$

Properties of Orthogonal and Orthonormal Basis

Theorem

1. *If* $\{\vec{v}_1, \ldots, \vec{v}_k\}$ *is an orthonormal basis for a subspace* W *and* $\vec{w} \in W$ *then*

$$proj_W \vec{x} = (\vec{x} \cdot \vec{v}_1)\vec{v}_1 + \cdots + (\vec{x} \cdot \vec{v}_k)\vec{v}_k$$

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$$[w]_{\mathcal{B}} = \begin{pmatrix} w \cdot v_1 \\ \vdots \\ w \cdot v_k \end{pmatrix}$$

v_1-coordinate of $\vec{w}$ in $\mathcal{B}$

v_k-coordinate of $\vec{w}$ in $\mathcal{B}$

v_i-coordinate of $\bar{w}$ in $\mathcal{B}$ is $(w \cdot v_i)$

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If $\mathcal{B}$ is orthonormal then $\|\vec{v}_i\|^2 = 1$

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Niceness of Orthonormal Basis

In particular, this theorem states that if $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthonormal basis for $\vec{w} \in \mathbb{R}^n$, then,

$$[\vec{w}]_B$$

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and hence if $B' = \{\vec{v}'_1, \ldots, \vec{v}'_n\}$ is another basis, then finding the transition matrix $P_{B' \to B}$ is easy:

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and hence if $B' = \{\vec{v}_1', \dots, \vec{v}_n'\}$ is another basis, then finding the transition matrix $P_{B' \to B}$ is easy:

$$P_{B' \to B} = \begin{pmatrix} [\vec{v}_1']_B & \dots & [\vec{v}_n']_B \end{pmatrix}$$

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$$P_{B \to B'} = (P_{B' \to B})^{-1}$$

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NOTE: it was imperative that we took B to be an *orthonormal* basis. This does NOT hold in general!

Gram-Schmidt Process

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Gram-Schmidt Process

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Suppose we have a basis $B = \{\vec{v}_1, \ldots, \vec{v}_k\}$ of a subspace W of R^n the algorithm on the next slide creates a new set of vectors $\{\vec{w}_1, \ldots, \vec{w}_k\}$ that is an *orthogonal* basis for W

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Gram-Schmidt Algorithm

1. Set

$$\vec{w}_1 = \vec{v}_1$$

Gram-Schmidt Algorithm

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2. Set

$$\vec{w}_2 = \vec{v}_2$$

Want w_2 to be orthogonal to w_1

$$V_2 = \text{proj}_{w_1} v_2 + \underline{\text{proj}_{w_1^\perp} V_2}$$

$V_2 - \text{proj}_{w_1} V_2$ is orthogonal to w_1

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$$\vec{w}_2 = \vec{v}_2 - \mathrm{proj}_{\mathrm{span}\{\vec{w}_1\}} \vec{v}_2$$

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3. Set

$$\vec{w}_3 = \vec{v}_3$$

$V_3 = \text{proj}_{\text{span}(w_1, w_2)} V_3 + \text{proj}_{\text{span}(w_1, w_2)^\perp} V_3$

$V_3 = \text{proj}_{\text{span}(v_1, w_2)} V_3$ is orthogonal to both

w_1 & w_2

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4. Continue the process to get $\vec{w}_1, \ldots, \vec{w}_k$.

$w_j = v_j - \mathrm{proj}_{\mathrm{span}\{w_1 \ldots w_{j-1}\}} v_j \in \mathrm{span}(w_1 \ldots w_{j-1})^{\perp}$

will be orthogonal to $w_1, \ldots w_j$

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$$\vec{u}_i = \frac{1}{\|\vec{w}_i\|}\vec{w}_i$$

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then $\{\vec{u}_1, \ldots, \vec{u}_k\}$ will be an orthonormal basis.