SF 1684 Algebra and Geometry
Lecture 14

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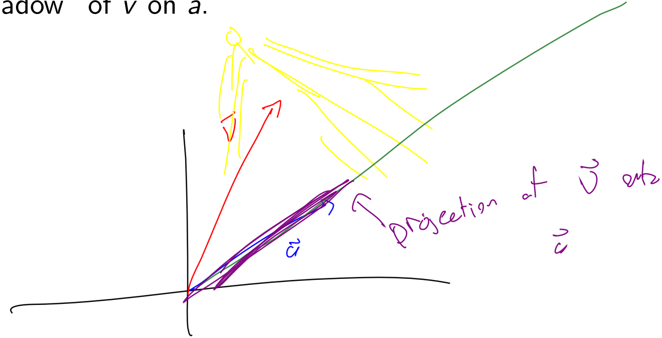
Topics for Today

1. Orthogonal Projections onto a Line
2. Orthogonal Projections onto a Subspace
3. Projection Matrices

Projections

Recall in Lecture 2 we defined the projection of a vector $\vec{v}$ onto another vector $\vec{a}$ as the "shadow" of $\vec{v}$ on $\vec{a}$.



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$$\text{proj}_{\vec{a}}\vec{v} = \frac{\vec{v} \cdot \vec{a}}{\|\vec{a}\|^2}\vec{a}$$

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scalar

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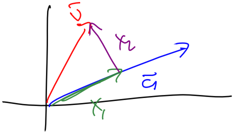

$$\mathrm{proj}_{\vec{a}}\vec{v} = \frac{\vec{v}\cdot\vec{a}}{\|\vec{a}\|^2}\vec{a}$$

Theorem

If $\vec{a}$ is a non-zero vector in $\mathbb{R}^n$, then every vector $\vec{x} \in \mathbb{R}^n$ can be expressed in exactly one way as

$$\vec{x} = \vec{x}_1 + \vec{x}_2$$

where $\vec{x}_1$ is a scalar multiple of $\vec{a}$ and $\vec{x}_2$ is orthogonal to $\vec{a}$ (and hence to $\vec{x}_1$).

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where $\vec{x}_1$ is a scalar multiple of $\vec{a}$ and $\vec{x}_2$ is orthogonal to $\vec{a}$ (and hence to $\vec{x}_1$).In particular, we have

$$\vec{x}_1 = proj_{\vec{a}}\vec{x}_1 = \frac{\vec{x} \cdot \vec{a}}{\|\vec{a}\|^2}\vec{a} \qquad \vec{x}_2 = \vec{x} - \vec{x}_1 = \vec{x} - \frac{\vec{x} \cdot \vec{a}}{\|\vec{a}\|^2}\vec{a}$$

Proof

$\vec{x}_1 = c \cdot \vec{a}$ $\qquad$ $\vec{x} = \vec{x}_1 + \vec{x}_2 \to \vec{x}_2 = \vec{x} - \vec{x}_1 = x - c \cdot \vec{a}$

We also need $\vec{x}_2 \cdot \vec{a} = 0$

that is, need $(\vec{x} - c \cdot \vec{a}) \cdot \vec{a} = 0$

Expand this. $\vec{x} \cdot \vec{a} - c(\vec{a} \cdot \vec{a}) = 0$

Rearranging, find $c = \dfrac{\vec{x} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} = \dfrac{\vec{x} \cdot \vec{a}}{\|\vec{a}\|^2}$

Hence we can conclude that $\vec{x}_1 = \dfrac{\vec{x} \cdot \vec{a}}{\|\vec{a}\|^2} \cdot \vec{a} = \mathrm{proj}_{\vec{a}} \vec{x}$

Orthogonal Projections and Components

While $\vec{a}$ is a vector span$(\vec{a})$ is a subspace.

Definition

If $\vec{a}$ is a nonzero vector in $\mathbb{R}^n$ and if $\vec{x}$ is any vector in $\mathbb{R}^n$, then the **orthogonal projection of $\vec{x}$ onto span**$(\vec{a})$ is denoted $\text{proj}_{\vec{a}}\vec{x}$ and defined to be

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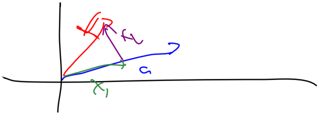
The vector $\text{proj}_{\vec{a}}\vec{x}$ is also called the **vector component of** $\vec{x}$ **along** $\vec{a}$ and $\vec{x} - \text{proj}_{\vec{a}}\vec{x}$ is called the **vector component of** $\vec{x}$ **orthogonal to** $\vec{a}$.



$\vec{x} = \vec{x}_1 + \vec{x}_2$

Example

Let $\vec{x} = (2, -1, 3)$ and $\vec{a} = (4, -1, 2)$. Find the vector component of $\vec{x}$ along $\vec{a}$ and the vector component of $\vec{x}$ orthogonal to $\vec{a}$.

$$x_1 = \operatorname{proj}_{\vec{a}} \vec{x} = \frac{x \cdot a}{\|a\|^2} \cdot a = \frac{13}{20}\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 13/5 \\ -13/20 \\ 13/10 \end{pmatrix}$$

← vector component of x along $\vec{a}$

$$x \cdot a = 2 \cdot 4 + -1 \times -1 + 2 \times 3$$
$$= 8 + 1 + 6$$
$$= 13$$

$$\|a\|^2 = a \cdot a = 4 \times 4 + 1 \times 1 + 2 \times 2$$
$$= 16 + 1 + 4$$
$$= 20$$

error

$$x_2 = x - x_1$$
$$= \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 13/5 \\ -13/20 \\ 13/10 \end{pmatrix}$$
$$= \begin{pmatrix} -3/5 \\ -7/20 \\ 17/10 \end{pmatrix}$$

component of x_2 orth to $\vec{a}$

Exercise

check: $\vec{x}_2 \cdot \vec{a} = 0$

Orthogonal Projections as Linear Transformations

For any vector $\vec{a} \in \mathbb{R}^n$, we can define the map $T : \mathbb{R}^n \to \mathbb{R}^n$

$$T(\vec{x}) = \mathrm{proj}_{\vec{a}}\vec{x} = \frac{\vec{x} \cdot \vec{a}}{\|\vec{a}\|^2}\vec{a}$$

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Exercise

Show that T is a linear transformation.

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Exercise

Show that T is a linear transformation.

We call this map the **orthogonal projection of** $\mathbb{R}^n$ **onto span**$(\vec{a})$.

subspace.

Standard Matrix of Orthogonal Projection

Theorem

If $\vec{a}$ is a nonzero vector in $\mathbb{R}^n$, and if $\vec{a}$ is viewed as an $n \times 1$ matrix, then the standard matrix for the linear operator $T(\vec{x}) = proj_{\vec{a}}\vec{x}$ is

$$P = \overbrace{\frac{1}{\vec{a}^T\vec{a}}}^{\text{Scalar}} \vec{a}\vec{a}^T \text{ - Square matrix}$$

Note: $\underbrace{\vec{a}^T\vec{a}}_{\vec{a}\cdot\vec{a}} \in \mathbb{R}^1$ and so is a scalar, whereas $\vec{a}\vec{a}^T$ is an $n \times n$ matrix.

Proof: $a^T a = a \cdot a = \|a\|^2$

$$\vec{a}\vec{a}^T = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} = \begin{bmatrix} a_1a_1 & a_2a_1 & \cdots & a_na_1 \\ a_1a_2 & a_2a_2 & \cdots & a_na_2 \\ \vdots & \vdots & & \vdots \\ a_1a_n & a_2a_n & \cdots & a_na_n \end{bmatrix}$$

More Work Space

$$P = \frac{1}{a^T a} a a^T = \frac{1}{\|a\|^2}\begin{bmatrix} a_1 a_1 & \cdots & a_n a_1 \\ \vdots & & \vdots \\ a_1 a_n & \cdots & a_n a_n \end{bmatrix}$$

$$T(\vec{x}) = \text{proj}_a \vec{x} = \frac{a \cdot x}{\|a\|^2} \cdot a.$$

Recall: the i^{th} column of the standard matrix of T will be $T(\vec{e}_i)$

$$T(e_i) = \frac{a \cdot e_i}{\|a\|^2} \cdot \vec{a}$$

$$a \cdot e_i = a_1 \cdot 0 + a_2 \cdot 0 + \cdots + a_i \cdot 1 + \cdots + 0 \cdot e_n = a_i$$

$$= \frac{a_i}{\|a\|^2}\vec{a} = \frac{a_i}{\|a\|^2}\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \frac{1}{\|a\|^2}\begin{bmatrix} a_i \cdot a_1 \\ \vdots \\ a_i \cdot a_n \end{bmatrix}$$

Example

Find the standard matrix of the linear transformation given by projecting onto span$\{(4, -1, 2)\}$.

$$\vec{a} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$$

$$\vec{a}^T \vec{a} = a \cdot a = 4\times 4 + -1\times -1 + 2\times 2 = 16 + 1 + 4 = 21$$

$$a a^T = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \begin{bmatrix} 4 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 4\times 4 & -1\times 4 & 2\times 4 \\ 4\times -1 & -1\times -1 & 2\times -1 \\ 4\times 2 & -1\times 2 & 2\times 2 \end{bmatrix} = \begin{bmatrix} 16 & -4 & 8 \\ -4 & 1 & -2 \\ 8 & -2 & 4 \end{bmatrix}$$

$$\mathrm{proj}_{\vec{a}} \vec{x} = \frac{1}{21} \begin{bmatrix} 16 & -4 & 8 \\ -4 & 1 & 2 \\ 8 & -2 & 4 \end{bmatrix} \vec{x}$$

Exercise: Check that this gives the same answer as the previous method.

Projection Theorem for Subspaces

So far we have talked about projecting onto a line given by span$\{\vec{a}\}$.

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Theorem

Let W be a subspace of $\mathbb{R}^n$, then every vector $\vec{x} \in \mathbb{R}^n$ can be expressed in exactly one way as

$$\vec{x} = \vec{x}_1 + \vec{x}_2$$

where $\vec{x}_1 \in W$ and $\vec{x}_2 \in W^\perp$.

$W^\perp = \{ v \in \mathbb{R}^n : \quad v \cdot w = 0 \quad \text{for all} \quad w \in W \}$

if $W = \text{span}\{\vec{a}\} \Rightarrow W^\perp = \{ v \in \mathbb{R}^n : \quad v \cdot \vec{a} = 0 \}$

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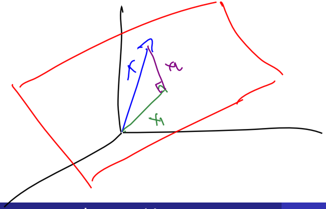
We call $\vec{x}_1$ the **orthogonal projection of $\vec{x}$ onto** W



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We call $\vec{x}_1$ the **orthogonal projection of $\vec{x}$ onto W** and $\vec{x}_2$ the **orthogonal projection of $\vec{x}$ on $W^\perp$** and denote them

$$\vec{x}_1 = \mathrm{proj}_W \vec{x} \qquad \text{and} \qquad \vec{x}_2 = \mathrm{proj}_{W^\perp} \vec{x}$$

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We can prove this theorem by constructing a value for $\mathrm{proj}_W \vec{x}$ that works.

Orthogonal Projection onto W

Theorem

If W is a nonzero subspace of $\mathbb{R}^n$, and if M is any matrix whose column vectors form a basis for W, then setting

$$\vec{x}_1 = proj_W \vec{x} = \underline{M(M^T M)^{-1} M^T} \vec{x}$$

satisfies the previous theorem.

If $W = \text{span}\{\vec{a}\}$ $\quad M = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

$$\begin{aligned} & M (M^T M)^{-1} M^T \\ &= \vec{a} (\vec{a}^T \vec{a})^{-1} \vec{a}^T \\ &= \vec{a} \left(\frac{1}{a^T a} \right) \vec{a}^T \\ &= \frac{1}{a^T a} \vec{a} \vec{a}^T \end{aligned}$$

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$$proj_W \vec{x} \in W \qquad \text{and} \qquad \vec{x}_2 = \vec{x} - \vec{x}_1 = \vec{x} - proj_W \vec{x} \in W^{\perp}$$

Proof.

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Proof.

See page 384 of textbook. □

$(x - proj_W x) \cdot v = 0$
for all $v \in W$.

Example

Let $\vec{x} = (1, 0, 4) \in \mathbb{R}^3$. Find the orthogonal projection of $\vec{x}$ onto the plane $P : x - 4y + 2z = 0$ as well the orthogonal projection onto $P^{\perp}$.

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First, let us find a basis for P: if $\vec{x} = (x, y, z) \in P$ then $x = 4y \overset{-}{\cancel{+}} 2z$

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First, let us find a basis for P: if $\vec{x} = (x, y, z) \in P$ then $x = 4y \cancel{+} - 2z$

$y = s$

$z = t$

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4s \cancel{+} - 2t \\ s \\ t \end{bmatrix}$$

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$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4s \cancel{+} - 2t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} t$$

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Thus we see a basis for P is

$$\left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Example Continued

Thus, forming the matrix M whose columns are the basis for P, we see that

$$M = \begin{pmatrix} 4 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example Continued

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$$M = \begin{pmatrix} 4 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad M^T = \begin{pmatrix} 4 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix}$$

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$$(M^T M)^{-1} = \begin{pmatrix} 5/21 & 8/21 \\ 8/21 & 17/21 \end{pmatrix}$$

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Hence the standard matrix for the orthogonal projection onto P will be

$$A = M(M^T M)^{-1} M^T$$

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Hence the standard matrix for the orthogonal projection onto P will be

$$A = M(M^T M)^{-1} M^T = \begin{pmatrix} 20/21 & 4/21 & -2/21 \\ 4/21 & 5/21 & 8/21 \\ -2/21 & 8/21 & 17/21 \end{pmatrix}$$

Example Continued

Therefore, the orthogonal projection of $\vec{x} = (1, 0, 4)$ onto the plane will be

$$\text{proj}_P \vec{x} = A\vec{x}$$

Example Continued

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Example Continued

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$$\mathrm{proj}_P \vec{x} = A\vec{x} = \begin{pmatrix} 20/21 & 4/21 & -2/21 \\ 4/21 & 5/21 & 8/21 \\ -2/21 & 8/21 & 17/21 \end{pmatrix} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 4/7 \\ 12/7 \\ 22/7 \end{bmatrix}$$

Exercise: check that $\begin{pmatrix} 4/7 \\ 12/7 \\ 22/7 \end{pmatrix}$ is on the plane

Example Continued

Therefore, the orthogonal projection of $\vec{x} = (1, 0, 4)$ onto the plane will be

$$\text{proj}_P \vec{x} = A\vec{x} = \begin{pmatrix} 20/21 & 4/21 & -2/21 \\ 4/21 & 5/21 & 8/21 \\ -2/21 & 8/21 & 17/21 \end{pmatrix} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 4/7 \\ 12/7 \\ 22/7 \end{bmatrix}$$

Moreover, $\text{proj}_{P^\perp} \vec{x} = \vec{x} - \vec{x}_1$

$x = x_1 + x_2$

$x_1 = \text{proj}_P x$, $x_2 = \text{proj}_{P^\perp} x$

Example Continued

Therefore, the orthogonal projection of $\vec{x} = (1, 0, 4)$ onto the plane will be

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Moreover, $\text{proj}_{P^\perp} \vec{x} = \vec{x} - \vec{x}_1$ and so

$$\text{proj}_{P^\perp} \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} - \begin{bmatrix} 4/7 \\ 12/7 \\ 22/7 \end{bmatrix} = \begin{bmatrix} 3/7 \\ -12/7 \\ 6/7 \end{bmatrix}$$

a normal for the plane

Example Continued

Therefore, the orthogonal projection of $\vec{x} = (1, 0, 4)$ onto the plane will be

$$\text{proj}_P \vec{x} = A\vec{x} = \begin{pmatrix} 20/21 & 4/21 & -2/21 \\ 4/21 & 5/21 & 8/21 \\ -2/21 & 8/21 & 17/21 \end{pmatrix} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 4/7 \\ 12/7 \\ 22/7 \end{bmatrix}$$

Moreover, $\text{proj}_{P^\perp} \vec{x} = \vec{x} - \vec{x}_1$ and so

$$\text{proj}_{P^\perp} \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} - \begin{bmatrix} 4/7 \\ 12/7 \\ 22/7 \end{bmatrix} = \begin{bmatrix} 3/7 \\ -12/7 \\ 6/7 \end{bmatrix}$$

Exercise

Show that $(3/7, -12/7, 6/7) \in P^\perp$.

Hint: enough to check it is orthogonal with the basis vectors.

Some Remarks

We can think of $\text{proj}_W \vec{x}$ as the "component of $\vec{x}$ that lies in W".

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$$x = \underset{\substack{\Uparrow \\ W}}{x_1} + \underset{\substack{\Uparrow \\ W^\perp}}{x_2}$$ unique

if $x \in W \Rightarrow x = \underset{\substack{\Uparrow \\ W}}{x} + \underset{\substack{\Uparrow \\ W^\perp}}{0}$

Since unique get $x = \text{proj}_W x$

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Example Redux

Back to the example before, we saw that the standard matrix for proj_P will be

$$A = \begin{pmatrix} 20/21 & 4/21 & -2/21 \\ 4/21 & 5/21 & 8/21 \\ -2/21 & 8/21 & 17/21 \end{pmatrix}$$

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And so the standard matrix for $\text{proj}_{P^\perp}$ would be

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Exercise

Confirm the previous example by showing that

$\text{proj}_{P^\perp}\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$

$$\begin{pmatrix} 1/21 & -4/21 & 2/21 \\ -4/21 & 16/21 & -8/21 \\ 2/21 & -8/21 & 4/21 \end{pmatrix} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 3/7 \\ -12/7 \\ 6/7 \end{bmatrix}$$

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Hence we see in the case where M is a square $n \times n$ matrix, we get that

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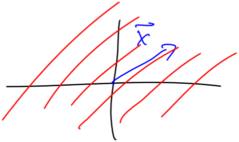
Hence, $\text{proj}_W \vec{x}$ is the "component of $\vec{x}$ lying in $W = \mathbb{R}^n$", which would be just $\vec{x}$ itself.

$X \in \mathbb{R}^n$

$n = 2$

$W = \mathbb{R}^2$



Double Perp Theorem

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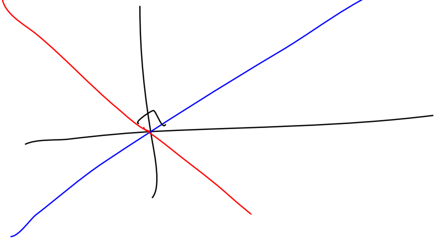
Theorem (Double Perp Theorem)

If W is a subspace of $\mathbb{R}^n$ then $(W^\perp)^\perp = W$

$W^\perp = \text{span}(\vec{w})$

$W = \text{span}(\vec{v}) = (W^\perp)^\perp$



$(W^\perp)^\perp = \{\text{orthogonal to } \vec{w}\}$

$= \{\text{direction of } \vec{v}\}$

We can use this notion to prove the **double perp theorem**.

Theorem (Double Perp Theorem)

If W is a subspace of $\mathbb{R}^n$ then $(W^\perp)^\perp = W$, i.e. "the perp space of the perp space is the original space."

proj onto W.

$x = x_1 + x_2$ uniquely where $x_1 \in W$, $x_2 \in W^\perp$

in particular $x_1 = \mathrm{proj}_W x$, $x_2 = \mathrm{proj}_{W^\perp} x$

projection onto $W^\perp$

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in particular $y_1 = \mathrm{proj}_{W^\perp} x$, $y_2 = \mathrm{proj}_{(W^\perp)^\perp} x$

$= x_2$

$= x - y_1$

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$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & & \\ a_{31} & a_{32} & \ddots & & \\ \vdots & & & \ddots & \end{bmatrix} \qquad = M(M^T M)^{-1} M^T = A$$

Definition

We say a matrix A is **symmetric** if $A^T = A$. Equivalently, its "upper triangle" is the same as its "lower triangle".

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In particular, this shows that $T \circ T = T$. Hence, if A is the standard matrix of T, this corresponds to saying

Stand mtr $T \circ T$ is A^2

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Definition

We say a matrix is **idempotent** if $A^2 = A$.

Exercise

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Show that if $A = M(M^T M)^{-1} M^T$ for some matrix M then $A^2 = A$.

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Show that the matrices of proj_P and $\text{proj}_{P^\perp}$ from the previous example are idempotent and symmetric. That is, if

$$A := \begin{pmatrix} 20/21 & 4/21 & -2/21 \\ 4/21 & 5/21 & 8/21 \\ -2/21 & 8/21 & 17/21 \end{pmatrix} \qquad B := \begin{pmatrix} 1/21 & -4/21 & 2/21 \\ -4/21 & 16/21 & -8/21 \\ 2/21 & -8/21 & 4/21 \end{pmatrix}$$

then $A^T = A$, $B^T = B$, $A^2 = A$ and $B^2 = B$.

Projection Matrices Theorem

Theorem

An $n \times n$ matrix A is the standard matrix for an orthogonal projection of $\mathbb{R}^n$ onto a k-dimensional subspace of $\mathbb{R}^n$ if and only if A is symmetric, idempotent and has rank k. The subspace, W, that A projects onto is then the column space of A.