SF 1684 Algebra and Geometry
Lecture 13

Patrick Meisner

KTH Royal Institute of Technology

Topics for Today

1. Dimension Theorem
2. Rank Theorem
3. Pivot Theorem

Nullity and Dimension Theorem

Nullity and Dimension Theorem

Definition

We will define the **nullity** of matrix A to be $\text{nullity}(A) = \dim(\text{null}(A))$.

nullity$(A) = \dim(\text{null}(A)) = \#$ free variables in RREF of A.

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Theorem (Dimension Theorem)

Let A be an $m \times n$ matrix, then

$$rk(A) + nullity(A) = n = \# \text{ of columns of } A.$$

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Let A be an $m \times n$ matrix, then

$$rk(A) + nullity(A) = n$$

Proof.

Indeed, we know that

$$\begin{aligned} n &= \text{number of columns} \\ &= \text{number of leading ones} + \text{number of free variable} \\ &= \text{rk}(A) + \dim(\text{null}(A)) = rk(A) + nullity(A) \end{aligned}$$

Perp Space

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Theorem

For any subspace W of $\mathbb{R}^n$, $W^\perp$ is also a subspace of $\mathbb{R}^n$.

Proof: $u, v \in W^\perp$ $\quad u+v \in W^\perp \iff (u+v)\cdot \vec{w} = 0$ for all $w \in W$

$(u+v)\cdot w = u\cdot w + v\cdot w = 0 + 0 = 0$ for all $w \in W$ & so $u+v \in W^\perp$

$u \in W^\perp$ & $c \in \mathbb{R}$ then $cu \in W^\perp \iff (cu)\cdot w = 0$ for all $w \in W$

$(cu) w = c(u \cdot v) = c(0) = 0$ for $v \in W$ & so $cu \in W^\perp$

Examples

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$$0 = (\vec{a}t) \cdot \vec{x}$$

for all $t \in \mathbb{R}$

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$$0 = (\vec{a}t) \cdot \vec{x} = a_1 t x_1 + a_2 t x_2 + \cdots + a_n t x_n$$

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

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$$0 = (\vec{a}t) \cdot \vec{x} = a_1tx_1 + a_2tx_2 + \cdots + a_ntx_n = t(a_1x_1 + a_2x_2 + \cdots + a_nx_n)$$

for all $t \in \mathbb{R}$.

for $t=1$: $a_1x_1 + \cdots + a_nx_n = 0$ $*$

$\Uparrow$

equation for hyperplane with normal $\vec{a}$.

Moreover if $*$ then $(\vec{a}\,t)\,\vec{x} = 0$ for all t.

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$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \quad \cdots$$

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$$0 = \vec{e_i} \cdot \vec{x} = 0 * x_1 + 0 * x_2 + \cdots + \underline{1 * x_i} + \cdots + 0 * x_n = x_i \quad \text{For all } i$$

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It is enough to just check $\vec{a} \cdot x = 0$

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$$0 = \vec{e_i} \cdot \vec{x} = 0 * x_1 + 0 * x_2 + \cdots + 1 * x_i + \cdots + 0 * x_n = x_i$$

And we see that $(\mathbb{R}^n)^\perp = \{\vec{0}\}$, the zero-suspace. if $\vec{x} = \vec{0}$ the $x \cdot v = 0$ for all vectors

Perp Space and Bases

Theorem

If W is a subspace of $\mathbb{R}^n$ with basis $\{\vec{b}_1, \ldots, \vec{b}_k\}$, then $\vec{v} \in W^\perp$ if and only if $\vec{v} \cdot \vec{b}_i = 0$ for $i = 1, \ldots, k$.

Proof: $(\Longrightarrow)$ if $v \in W^\perp$ the $v \cdot w = 0$ for $w \in W$

in particular $v \cdot b_i = 0$ for all i as $b_i \in W$.

$(\Longleftarrow)$ Suppose $\vec{v} \cdot \vec{b}_i = 0$ for all $i = 1, \ldots, k$. let $w \in W$ the can write $w = t_1 b_1 + \cdots + t_k b_k$.

$$v \cdot w = v \cdot (t_1 b_1 + \cdots + t_k b_k) = t_1 (v_1 \cdot b_1) + \cdots + t_k (v_k \cdot b_k)$$

$$= t_1 \cdot 0 + \cdots + t_k \cdot 0 = 0$$

& so $v \in W^\perp$

Perp Space and Null Space

Corollary

If W is a subspace of $\mathbb{R}^n$ with basis $\{\vec{b}_1, \ldots, \vec{b}_k\}$, then $W^\perp$ is the null space of the matrix whose rows are the $\vec{b}_i$.

Proof: $B = \begin{bmatrix} \vec{b}_1 \\ \vdots \\ \vec{b}_k \end{bmatrix}$

$x \in \text{null}(B) \iff B\vec{x} = \vec{0}$

$\iff \begin{bmatrix} \vec{b}_1 \\ \vdots \\ \vec{b}_k \end{bmatrix} \vec{x} = \vec{0}$

$\iff \begin{bmatrix} \vec{b}_1 \cdot \vec{x} \\ \vdots \\ \vec{b}_k \cdot x \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

$\iff \vec{b}_i \cdot \vec{x} = 0$ for all i.

$\iff x \in W^\perp$

Aside: Commonly we want to consider the matrix A with the columns of the basis vectors $A = [\vec{b}_1, \ldots, \vec{b}_k]$. But notice $B = A^T$. Then we conclude that $W^\perp = \text{null}(A^T)$

Perp Space and Dimension Theorem

Theorem

If W is a subspace of $\mathbb{R}^n$, then $n = \dim(W) + \dim(W^{\perp})$.

Proof: suppose W had a basis $b_1 \dots b_k$

let $B = \begin{bmatrix} b_1 \\ \vdots \\ b_k \end{bmatrix}$. I know that since $b_1 \dots b_k$ are linearly independent, then every row will have a leading 1.

$B \to \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \Rightarrow rk(B) = k = \dim W$

from previous $\dim W^{\perp} = \text{nullity}(B)$

Dimension thm: $n = rk(B) + \text{nullity}(B) = \dim W + \dim W^{\perp}$.

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Recall the column space of a matrix col(A), is the spanning set of the column vectors of A.

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If A is an $m \times n$ matrix, then

$\mathrm{col}(A^T) = \mathrm{row}(A)$

$$rk(A) = rk(A^T)$$

$rk(A) = \dim(\mathrm{col}(A)) \Rightarrow$

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In particular this shows that

$$\dim(\text{row}(A)) = \dim(\text{col}(A^T)) \leq \dim(\text{row}(A^T)) = \dim(\text{col}(A))$$

Now I have $\dim(\text{col}(A)) \leq \dim(\text{row}(A)) \leq \dim(\text{col}(A))$

So they must be equal

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Let A be an $m \times n$ matrix. Then we know that $\text{col}(A)$ is a subspace of $\mathbb{R}^m$ and so $\dim(\text{col}(A)) \leq m$.

$A = \begin{pmatrix} c_1 & \cdots & c_n \end{pmatrix}$ $\qquad c_i \in \mathbb{R}^m$

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Moreover, we know that $\text{row}(A)$ is a subspace of $\mathbb{R}^n$ spanned by m vectors and so $\dim(\text{row}(A)) \leq m$.

$$A = \begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix}$$

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If $\dim(\text{row}(A)) < m$, then one row of A must be written as a linear combination of the rest. That is, there exists $c_1, \ldots, c_{m-1}$ such that

$$\vec{r}_m = c_1\vec{r}_1 + \cdots c_{m-1}\vec{r}_{m-1}$$

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We have

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$\vec{r}_m = \begin{bmatrix} a_{m,1} \\ \vdots \\ a_{m,n} \end{bmatrix}$

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Looking at the j^{th} entry of the row vectors, we get

$$a_{m,j} = c_1 a_{1,j} + \cdots + c_{m-1} a_{m-1,j}$$

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can be written as a linear combination of $m-1$ vectors and so $\dim(\mathrm{col}(A)) \leq m-1$.

$\mathrm{col}(A) \subseteq \mathrm{span}(v_1 \ldots v_{m-1})$

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$$\dim(col(A)) = k \qquad \dim(null(A^T)) = m - k$$

Consistency and Rank

Theorem

If $A\vec{x} = \vec{b}$ is a linear system of m equations in n unknowns, then the following statements are equivalent

1. $A\vec{x} = \vec{b}$
2. *$\vec{b}$ is in the column space of A*
3. *The coefficient matrix A and the augmented matrix $(A|\vec{b})$ have the same rank.*

Full Rank

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Theorem

Let A be an $m \times n$ matrix.

1. *A has full column rank if and only if the column vectors form a basis for the column space if and only if $rk(A) = n$*
2. *A has full row rank if and only if the row vectors form a basis for the row space if and only if $rk(A) = m$*

Full Column Rank and Solutions

Theorem

If A is an $m \times n$ matrix then the following are equivalent

1. $A\vec{x} = 0$ *has only the trivial solution*
2. $A\vec{x} = \vec{b}$ *has at most one solution for every* $\vec{b} \in \mathbb{R}^m$
3. A *has full column rank*

Over- and Underdetermined

Theorem

Let A be an $m \times n$ matrix.

1. *If $m > n$, then the system $A\vec{x} = \vec{b}$ is inconsistent for some vector $\vec{b}$ in $\mathbb{R}^m$. This is called* **overdetermined**.
2. *If $m < n$, then for every vector $\vec{b}$ in $\mathbb{R}^m$ the system $A\vec{x} = \vec{b}$ is either inconsistent or has infinitely many solutions. This is called* **underdetermined**.

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Example:

$$A = \begin{pmatrix} 7 & 1 & 9 & 6 \\ 5 & 3 & 11 & 2 \\ 7 & 2 & 11 & 5 \\ 7 & -2 & 3 & 9 \\ 0 & 1 & 2 & -1 \end{pmatrix}$$

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$$A = \begin{pmatrix} 7 & 1 & 9 & 6 \\ 5 & 3 & 11 & 2 \\ 7 & 2 & 11 & 5 \\ 7 & -2 & 3 & 9 \\ 0 & 1 & 2 & -1 \end{pmatrix} \implies \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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And so a basis for the column space of A would be

$$\left\{ \begin{bmatrix} 7 \\ 5 \\ 7 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ -2 \\ 1 \end{bmatrix} \right\}$$

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See example in the slides of last lecture.

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The last remaining fundamental space of a matrix is null(A^T). This can also be determined by find the REF.

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4. The row vectors of E_2 now form a basis for null(A^T).

Example

Find null(A^T) for

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & 4 & 2 & -5 & -4 \end{bmatrix}$$

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Step 1: Adjoin I_4 to A:

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Step 1: Adjoin I_4 to A:

$$(A|I_4) = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 & 1 & 0 & 0 & 0 \\ 2 & -6 & 9 & -1 & 8 & 2 & 0 & 1 & 0 & 0 \\ 2 & -6 & 9 & -1 & 9 & 7 & 0 & 0 & 1 & 0 \\ -1 & 3 & 4 & 2 & -5 & -4 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example Continued

Step 2: Row reduce A to REF

$$(U|E) = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & -2 & 6 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Example Continued

Step 2: Row reduce A to REF

$$(U|E) = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & -2 & 6 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Step 3: Repartition it into

$$\left(\begin{array}{c|c} V & E_1 \\ 0 & E_2 \end{array}\right) = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & -2 & 6 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Example Continued

Step 2: Row reduce A to REF

$$(U|E) = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & -2 & 6 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

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Hence, we see that $\{(1, 0, 0, 1)\}$ is a basis for null(A^T).