SF 1684 Algebra and Geometry
Lecture 12

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Topics for Today

1. Bases and Dimension
2. Building Bases out of Linearly Independent Sets
3. Building Bases out of Spanning Sets

Recollections

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For any set of vectors $\vec{v}_1, \ldots, \vec{v}_k$, we define

$$\operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_k\} = \{t_1\vec{v}_1 + \cdots + t_k\vec{v}_k : t_1, \ldots, t_k \in \mathbb{R}\}.$$

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For any vector space, V, we say that $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a spanning set of V if

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Moreover, we say that $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent if none of the vectors can be written as a linear combination of the others

Finally, we know this is equivalent to saying that every vector in the span of $\vec{v}_1, \ldots, \vec{v}_k$ can be written *uniquely* as a linear combination of the $\vec{v}_i$.

Basis

Definition

We say that a set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a **basis** for a vector space V if it is **linearly independent** and **spans** V.

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Example:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ is a basis for } \mathbb{R}^3$$

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Exercise: Show this is a basis

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\} \text{ is a basis for } \mathbb{R}^3$$

Existence of Bases

Theorem

If V is a non-zero subspace of $\mathbb{R}^n$ then V has a basis consisting of fewer than n vectors.

Proofs, let $\vec{v}_1 \in V$, $\vec{v}_1 \neq \vec{0}$. $\text{span}(\vec{v}_1) = \{ t v_1 ; t \in \mathbb{R} \}$, is a subspace of V.

If $V = \text{span}(\vec{v}_1)$ the $\{\vec{v}_1\}$ is a basis

if $V \neq \text{span}(\vec{v}_1)$ then there exits $\vec{v}_2 \in V$ such that $\vec{v}_2 \notin \text{span}(\vec{v}_1)$

then necessarily, v_1 & v_2 are lin independent.

if $V = \text{span}(v_1, v_2)$ then $\{ \vec{v}_1, \vec{v}_2 \}$ is a basis.

if $V \neq \text{span}(v_1, v_2)$ then find v_3, —.

We know that any set of $n+1$ vectors will be linearly dependent. So this process stops eventually.

From Linearly Independent Set to Basis

We saw in the proof that if we have a set of linearly independent vectors, then we can systematically add in vectors that are not already in the span to form a basis.

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Expand the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \right\}$$

to a basis for $\mathbb{R}^4$

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First, we note that the two vectors are linearly independent so this is good. So, we need to find a vector that is not in the span.

From Linearly Independent Set to Basis 2

A vector $\vec{b}$ will be in the span if and only if there is a t_1, t_2 such that

$$t_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + t_2 \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

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$$\left(\begin{array}{cc|c} 1 & 5 & b_1 \\ 2 & 6 & b_2 \\ 3 & 7 & b_3 \\ 4 & 8 & b_4 \end{array}\right) \implies \left(\begin{array}{cc|c} 1 & 5 & b_1 \\ 0 & -4 & b_2 - 2b_1 \\ 0 & 0 & \underline{b_3 - 2b_2 + b_1} \\ 0 & 0 & \underline{b_4 - 3b_2 + 2b_1} \end{array}\right) \begin{array}{l} \\ \\ \neq 0 \\ \neq 0 \end{array}$$

Find a $\vec{b}$ such that

or

From Linearly Independent Set to Basis 3

Hence to find something **not** in the span, it is enough to find some $\vec{b}$ such that either $b_3 - 2b_2 + b_1 \neq 0$ or $b_4 - 3b_2 + 2b_1 \neq 0$.

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$$\vec{b} = \begin{bmatrix} 9 \\ 10 \\ 11 \\ 13 \end{bmatrix}$$

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Do the same process but now with the set

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 9 \\ 4 \\ 5 \end{bmatrix} \right\}$$

.

From Linearly Independent Set to Basis 3

We see that, for example

$A = \begin{pmatrix} | & | & | \end{pmatrix}$

Find $\vec{b}$ s.t. $(A \mid \vec{b})$ is not consistent

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \notin \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 9 \\ 4 \\ 5 \end{bmatrix} \right\}$$

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$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \notin \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 9 \\ 4 \\ 5 \end{bmatrix} \right\}$$

And so

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 9 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

is a linearly independent set.

From Linearly Independent Set to Basis 4

Finally, we see that since the columns are linearly independent, the matrix

$$A = \begin{pmatrix} 1 & 5 & 2 & 0 \\ 2 & 6 & 9 & 1 \\ 3 & 7 & 4 & 0 \\ 4 & 8 & 5 & 0 \end{pmatrix}$$

is invertible.

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is invertible.Therefore, for ever $\vec{b} \in \mathbb{R}^4$, there is a solution to $A\vec{x} = \vec{b}$.

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is invertible.Therefore, for ever $\vec{b} \in \mathbb{R}^4$, there is a solution to $A\vec{x} = \vec{b}$. In other words every vector in $\mathbb{R}^4$ can be written as a linear combination of the vectors and so

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 9 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

spans $\mathbb{R}^4$ and is a basis.

From Spanning Set to Basis

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Exercise

Find a basis for the subspace of $\mathbb{R}^5$ given by

$$V := \operatorname{span}\underbrace{\left\{\begin{bmatrix}7\\5\\7\\7\\0\end{bmatrix}, \begin{bmatrix}1\\3\\2\\-2\\1\end{bmatrix}, \begin{bmatrix}9\\11\\11\\3\\2\end{bmatrix}, \begin{bmatrix}6\\2\\5\\9\\-1\end{bmatrix}\right\}}$$

Note:
the vector space we are considering is <u>not</u> $\mathbb{R}^5$.

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First we check to see if they are linearly independent

From Spanning Set to Basis 2

To do this, we put the vectors in a matrix and row reduce

$$\begin{pmatrix} 7 & 1 & 9 & 6 \\ 5 & 3 & 11 & 2 \\ 7 & 2 & 11 & 5 \\ 7 & -2 & 3 & 9 \\ 0 & 1 & 2 & -1 \end{pmatrix}$$

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$$\begin{pmatrix} 7 & 1 & 9 & 6 \\ 5 & 3 & 11 & 2 \\ 7 & 2 & 11 & 5 \\ 7 & -2 & 3 & 9 \\ 0 & 1 & 2 & -1 \end{pmatrix} \implies \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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$$\begin{pmatrix} 7 & 1 & 9 & 6 \\ 5 & 3 & 11 & 2 \\ 7 & 2 & 11 & 5 \\ 7 & -2 & 3 & 9 \\ 0 & 1 & 2 & -1 \end{pmatrix} \implies \overset{*}{\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence, we see that

$$* \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -t-s \\ -2t+s \\ t \\ s \end{bmatrix}$$

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$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -t-s \\ -2t+s \\ t \\ s \end{bmatrix} \implies c_1 \begin{bmatrix} 7 \\ 5 \\ 7 \\ 7 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 2 \\ -2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 9 \\ 11 \\ 11 \\ 3 \\ 2 \end{bmatrix} + c_4 \begin{bmatrix} 6 \\ 2 \\ 5 \\ 9 \\ -1 \end{bmatrix} = \vec{0}$$

From Spanning Set to Basis 3

Setting $t = 1$, $s = 0$ we get $c_1 = -1$, $c_2 = -2$, $c_3 = 1$, $c_4 = 0$ and

$$\begin{bmatrix} 9 \\ 11 \\ 11 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 7 \\ 7 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 3 \\ 2 \\ -2 \\ 1 \end{bmatrix}$$

$v_3 = v_1 + 2v_2$

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Hence, we may remove this vector without affecting the span.

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Hence, we may remove this vector without affecting the span. Further, setting $t = 0$ and $s = 1$, we get $c_1 = -1$, $c_2 = 1$, $c_3 = 0$, $c_4 = 0$ and so

$$\begin{bmatrix} 6 \\ 2 \\ 5 \\ 9 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 7 \\ 7 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\vec{v}_4 = \vec{v}_1 - \vec{v}_2$$

From Spanning Set to Basis 3

Setting $t = 1$, $s = 0$ we get $c_1 = -1$, $c_2 = -2$, $c_3 = 1$, $c_4 = 0$ and

If we have many free variables $t_1, \ldots t_k$

Setting $t_1 = 1$, $t_2 = \cdots = t_k = 0$ gives one linear dependence

Setting $t_1 = 0$, $t_2 = 1$, $t_3 \cdots t_k = 0$ gives another and so on.

$$\begin{bmatrix} 9 \\ 11 \\ 11 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 7 \\ 7 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 3 \\ 2 \\ -2 \\ 1 \end{bmatrix}$$

$v_3 = v_1 + 2v_2$

Hence, we may remove this vector without affecting the span. Further, setting $t = 0$ and $s = 1$, we get $c_1 = -1$, $c_2 = 1$, $c_3 = 0$, $c_4 = 0$ and so

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$v_4 = v_1 - v_2$

So can remove v_3 & v_4 without affecting the span.

Hence we may remove this vector as well.

From Spanning Set to Basis 4

Thus we may conclude that

$$V = \operatorname{span}\left\{\begin{bmatrix}7\\5\\7\\7\\0\end{bmatrix}, \begin{bmatrix}1\\3\\2\\-2\\1\end{bmatrix}, \begin{bmatrix}9\\11\\11\\3\\2\end{bmatrix}, \begin{bmatrix}6\\2\\5\\9\\-1\end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix}7\\5\\7\\7\\0\end{bmatrix}, \begin{bmatrix}1\\3\\2\\-2\\1\end{bmatrix}\right\} = U$$

From Spanning Set to Basis 4

$\dim(V) = 2$

$V \subseteq \mathbb{R}^5$

Thus we may conclude that

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Moreover, the latter two are linearly independent and so form a basis.

Big note: I am not claiming that they form a basis for $\mathbb{R}^5$!!!

Theorem

Let V be a subspace of $\mathbb{R}^n$. Then if $\{\vec{v}_1, \ldots, \vec{v}_k\}$ and $\{\vec{w}_1, \ldots, \vec{w}_m\}$ are two bases for V then $k = m$. That is, the size of the basis is always the same.

Sketch of proof: $v_1, \ldots v_k \in V = \text{span}(w_1, \ldots w_m)$

$v_1 = a_{11}w_1 + \cdots + a_{1m}w_m$

$v_2 = a_{21}w_1 + \cdots + a_{2m}w_m$

$\vdots$

$v_k = a_{k1}w_1 + \cdots + a_{km}w_m$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{km} \end{pmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & & 0 & * \\ & \ddots & & \\ 0 & & 1 & \end{bmatrix}$$

Exercise: prove this claim.

[claim: $A\begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} = \vec{0} \implies c_1\vec{w}_1 + \cdots + c_m\vec{w}_m = 0$]

But $w_1 \ldots w_m$ are lin ind. $\implies c_1 = \cdots = c_m = 0$

Therefore, I deduce that the only homo solution to A is $\vec{0}$

$\implies k \le m$. Swapping the v's & w's gives $m \le k$ and so $k = m$

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If $V = \text{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$, then $\dim(V) \leq k$. If the $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent then $\dim(V) = k$.

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If L is a line, then $\dim(L) = 1$, since $L = \{t\vec{v} : t \in \mathbb{R}\} = \text{span}\{\vec{v}\}$.

Dimension

Definition

For any subspace V of $\mathbb{R}^n$, we define the **dimension of** V to be the number of vectors in any basis. We typically denote it $\dim(V)$.

Examples:

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Similarly, if P is a plane, then $\dim(P) = 2$.

Basis for the Zero-Subspace

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Definition

The basis of the zero subspace is the empty set.

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Definition

$$\dim(\{\vec{0}\}) = 0$$

Theorem

If V is a subspace of $\mathbb{R}^n$ then $\dim(V) = 0$ if and only if V is the zero subspace.

Dimension of Null Space

Recall that to find the subspace of homogeneous solutions (or null space) of a matrix A, we use Gauss-Jordan elimination and then find vectors so that the solution space is of the form $\text{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$ where k is the number of free variables.

$$\text{null}(A) = \{ \vec{x} : A\vec{x} = \vec{0} \}$$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & & a_{mn} \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & x & & x & x \\ & 0 & 1 & & x \\ & 0 & & & 1 \\ 0 & 0 & \cdots & & 0 \end{bmatrix}$$

$$\rightarrow V_1, V_2 \ldots V_k$$

k = # of leading 1s or $rk(A)$

of # of free variables

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Therefore, we can conclude that $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a basis for the null space of A and hence

$$\dim(\text{null}(A)) = k = \text{ number of free of variables } = nl(A)$$

Moreover, if $\vec{v}_1, \ldots, \vec{v}_k$ are obtained from the RREF of A, then we call this the **canonical basis** for the null space of A.

Maximal Linear Independence

Theorem

If V is a non-zero subspace of $\mathbb{R}^n$, then $\dim(V)$ is the maximum number of linearly independent vectors in V.

Recall, we already showed that any set of strictly more than n vectors in $\mathbb{R}^n$ will be linearly dependent. This is due to the fact that $\dim(\mathbb{R}^n) = n$.

Proof: Suppose $\dim V = k$ and we have $v_1 \dots v_m$ lin independent vectors in V with $m > k$. Then $\mathrm{span}(v_1, \dots, v_m)$ would be a subspace of V. Moreover, we have seen that we expand any lin independent set to a basis. That is find $w_1 \dots w_t$ such that the set $\{v_1 \dots v_m, w_1 \dots w_t\}$ is a basis for V. This implies that $\dim V = m + t > k$ contradiction.

Subsapces and Dimensions

Theorem

Let V and W be subspaces of $\mathbb{R}^n$. If V is a subspace of W, then

1. $0 \leq \dim(V) \leq \dim(W) \leq n$
2. *$V = W$ if and only if* $\dim(V) = \dim(W)$. $\Leftarrow$ necessary also to $V \subseteq W$. (very important)

Proof:

(1) Let $\{b_1 \dots b_k\}$ be a basis for V. Then in particular, $\{b_1 \dots b_k\}$ is a lin ind set in W. So we can expand this to a basis for W. & so $\dim W \geq k = \dim V$

(2) $(\Rightarrow)$ if $V = W$ then $\dim V = \dim W$

$(\Leftarrow)$ if $\dim V = \dim W$. Then if $\{b_1 \dots b_k\}$ is a basis for V. Then it is a lin independent subset in W. So we can expand to a basis. However, since $\dim W = \dim V = k$, we know that the expansion process would stop immediately. Hence $W = \mathrm{Span}(b_1, \dots, b_k) = V$.

Theorem

Theorem

Let V be a k-dimensional subspace of $\mathbb{R}^n$

1. *Any set of k linearly independent vectors of V is a basis for V (in particular, they span V)*
2. *Any set of k vectors that span V is a basis for V (in particular, they are linearly independent)*
3. *Any set of strictly fewer than k vectors of V cannot span ~~V~~ V*
4. *Any set of more than k vectors of V cannot be linearly independent*

first theorem

Proof: (1) let $(v_1 .. v_k)$ be a set of linearly independent vectors in V. $W = \text{span}(v_1 ... v_k)$. we know that $W \subseteq V$ & $\dim W = k = \dim V$

by previous thm: $W = V$ & so $V = \text{span}(v_1 ... v_k)$ & $v_1 ... v_k$ is a basis.

Proof

(2) Let $v_1 \dots v_k$ be a set of vectors of V such that $V = \text{span}(v_1 - v_k)$. Suppose that $v_1 \dots v_k$ is <u>not</u> lin ind. Then we can remove some of the v_i's and not change the span. That is we could say

$V = \text{span}(v_1 \dots v_{k-1}) \Rightarrow \dim V \le k-1$ which contradicts the assumption that $\dim V = k$.

(3) Suppose $v_1 \dots v_m$ spans V with $m < k$.
Then I can remove some to form a basis for V
But this implies that $\dim V \le m < k$
which contradicts the assumption that $\dim V = k$.

Corollary

1. *A set of n vectors in* $\mathbb{R}^n$ *is linearly independent if and only if they are a basis for* $\mathbb{R}^n$

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Proof.

Follows immediately from the previous theorem and the fact that $\dim(\mathbb{R}^n) = n$. □

Major Theorem

Theorem

Let A be an $n \times n$ matrix. The the following are equivalent

1. $A\vec{x} = \vec{b}$ *has a unique solution for every* $\vec{b}$
2. $A\vec{x} = 0$ *has a unique solution*
3. $rk(A) = n$
4. *The RREF of A is I_n*
5. *A is invertible*
6. *The columns of A are linearly independent*
7. *The rows of A are linearly independent*
8. $\det(A) \neq 0$
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