SF 1684 Algebra and Geometry
Lecture 11

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Topics for Today

1. Subspaces Associated to Linear Transformations: Kernel and Range
2. Compositions of Linear Transformations
3. Inverses of Linear Transformations

Kernel of a Linear Transformation

Definition

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then we say that the set of vectors in $\mathbb{R}^n$ that T maps to $\vec{0}$ is the **kernel** of T and denote it $\ker(T)$.

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Recall that for every such linear transformation, we can find an $m \times n$ matrix A such that $T(\vec{x}) = A\vec{x}$.

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$T(\vec{x}) = 0 \iff A\vec{x} = 0 \iff x$ is homs solution to A.

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$$\ker(T_A) = \text{null}(A)$$

Kernel of a Linear Transformation

If $T(\vec{x}) = 0$ the $\vec{x}$ is in the kernel of T

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Theorem

For any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, the kernel of T is a subspace of $\mathbb{R}^n$.

$\ker(T_A) = \text{null}(A) =$ homo solutions of A.

Exercise

Find the kernel of the linear transformations:

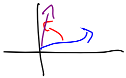
$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

1. $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ 3z \end{bmatrix}$
2. Rotating by an angle of θ in $\mathbb{R}^2$: T

(1) The kernel is all $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} x+y \\ 3z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{matrix} x+y=0 \\ 3z=0 \end{matrix} \Rightarrow \begin{matrix} x=-y \\ z=0 \end{matrix}$$

$$\ker(T) = \left\{ \begin{bmatrix} t \\ -t \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\} = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

(2) 

rotating by θ results in $\vec{0}$ if and only if you begin with $\vec{0}$

So $\ker(T) = \{\vec{0}\}$ = zero subspace.

One-to-one Linear Transformations

Definition

We say a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **one-to-one** (or **injective**) if T maps distinct vectors in $\mathbb{R}^n$ to distinct vectors in $\mathbb{R}^m$.

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$$\vec{x} \neq \vec{y} \implies T(\vec{x}) \neq T(\vec{y}) \qquad \text{or} \qquad T(\vec{x}) = T(\vec{y}) \implies \vec{x} = \vec{y}$$

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T not one-to-one $\implies \ker(T) \neq \{\vec{0}\}$

Theorem

$T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if and only if $\ker(T) = \{\vec{0}\}$, *the zero-subspace. For any matrix $m \times n$ matrix A, T_A is one-to-one if and only $A\vec{x} = \vec{0}$ has only the trivial solution.*

Suppose T is not one-to-one. Then there exists $\vec{x} \neq \vec{y}$ but $T(\vec{x}) = T(\vec{y})$

Therefore $T(\vec{x} - \vec{y}) = T(\vec{x}) - T(\vec{y}) = \vec{0} \implies \vec{x} - \vec{y} \in \ker(T)$

but $\vec{x} - \vec{y} \neq \vec{0}$

Range of a Linear Transformation

Definition

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then we say that the set of vectors in $\mathbb{R}^m$ that can be written in the form $T(\vec{x})$ is the **range** of T and denote it $\text{ran}(T)$.

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If A is the $m \times n$ matrix such that $T(\vec{x}) = A\vec{x}$ then $\mathrm{ran}(T)$ is the set of vectors $\vec{b}$ such that there is a solution to $A\vec{x} = \vec{b}$

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Theorem

For any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, the range of T is a subspace of $\mathbb{R}^m$.

$T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$ & $T(c\vec{x}) = cT(\vec{x})$

Exercise

$\exists x$ s.t. $T(\vec{x}) = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

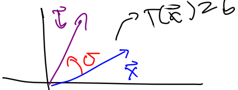
$\Updownarrow$

$\exists \vec{x}$ s.t. $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \vec{x} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$\Updownarrow$

Find the range of the linear transformation

1. $T(\vec{x}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \vec{x}$
2. Rotating by an angle of θ in $\mathbb{R}^2$



① $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ is in the range of T iff $\left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 1 & 1 & b_2 \end{array}\right)$ is consistent

$\left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 1 & 1 & b_2 \end{array}\right) \xrightarrow{R_2 - R_1} \left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 0 & \underline{b_2 - b_1} \end{array}\right)$ consistent iff $b_1 = b_2$

Range $= \left\{ \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \text{ s.t. } b_1 = b_2 \right\} = \left\{ \begin{bmatrix} t \\ t \end{bmatrix}, t \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

② Any vector can be obtained by a rotation by an angle of $\mathbb{R}^2$ so $\text{Ran}(T) = \mathbb{R}^2$

Onto Linear Transformations

Definition

We say a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **onto** (or **surjective**) if every vector in $\mathbb{R}^m$ is the image of at least one vector in $\mathbb{R}^n$

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Theorem

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if and only if $ran(T) = \mathbb{R}^m$. If A in an $m \times n$ matrix, then the linear transformation T_A is onto if and only $A\vec{x} = \vec{b}$ has a solution for all $\vec{b}$.

equivalently $(A \mid \bar{b})$ is consistent for all $\bar{b}$.

Pigeonhole Principle

Theorem

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one if and only if it is onto.

Proof: Let A be the standard matrix of T.

$T(\vec{x}) = A\vec{x}$. T is one-to-one $\iff$ $A\vec{x} = 0$ has only the trivial solution

$\iff$ $A\vec{x} = b$ has a solution for all $\vec{b}$

$\iff$ T is onto.

Composition of Linear Transformations

Definition

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^k$ be linear transformations. Then we say $S \circ T : \mathbb{R}^n \to \mathbb{R}^k$ is the **composition** of S and T and define it as

$$(S \circ T)(\vec{x}) = S\left(T(\vec{x})\right)$$

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$$(S \circ T)(\underset{\mathbb{R}^n}{\vec{x}}) = S\,(\underbrace{T(\vec{x})}_{\mathbb{R}^m})$$

NOTE: while $S \circ T$ exists, $T \circ S$ does not necessarily exist since S outputs vectors in $\mathbb{R}^k$ while T must have vectors in $\mathbb{R}^m$ input into it.

$\Rightarrow n = k$

even if $T \circ S$ & $S \circ T$ exists it does not follow that $T \circ S = S \circ T$

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NOTE: while $S \circ T$ exists, $T \circ S$ does not necessarily exist since S outputs vectors in $\mathbb{R}^k$ while T must have vectors in $\mathbb{R}^m$ input into it.

Theorem

$S \circ T$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^k$.

$$(S \circ T)(\vec{u}+\vec{v}) = S(T(\vec{u}+\vec{v})) \overset{T \text{ linear}}{=} S(T(\vec{u}) + T(\vec{v}))$$

$$S \text{ linear} \Rightarrow = S(T(u)) + S(T(v)) = (S \circ T)(\vec{u}) + (S \circ T)(\vec{v})$$

Inverse Transformations

Definition

We say that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is **invertible** if there is a linear transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$(\underline{S \circ T})(\vec{x}) = (\underline{T \circ S})(\vec{x}) = \vec{x}$$

for all $\vec{x} \in \mathbb{R}^n$.

$S \circ T$ & $T \circ S$ exist

$(S \circ T)(\vec{x}) = \vec{x}$

necessarily the output of $S \circ T$ must be the same space as the input.

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$$(S \circ T)\vec{x} = (T \circ S)\vec{x} = \vec{x}$$

for all $\vec{x} \in \mathbb{R}^n$. We call S the **inverse** of T and denote it T^{-1}.

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Note that we know that T_{I_n} has the property that $T_{I_n}(\vec{x}) = \vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

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for all $\vec{x} \in \mathbb{R}^n$. We call S the **inverse** of T and denote it T^{-1}.

Note that we know that T_{I_n} has the property that $T_{I_n}(\vec{x}) = \vec{x}$ for all $\vec{x} \in \mathbb{R}^n$. So we could rewrite the above statement as

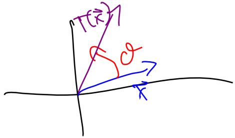
$$S \circ T = T \circ S = T_{I_n}$$

Exercise

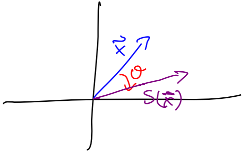
Exercise: Use the matrix to show algebraically that $S \circ T(\vec{x}) = \vec{x}$ for all $\vec{x}$

Find the inverse to the transformation obtained by rotation by an angle of θ.

T:



S:



The inverse operation would necessarily map $T(\vec{x})$ to $\vec{x}$

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The inverse operation is rotating in the opposite direction by an angle of θ. or can think about as rotating by an angle of $-\theta$.

$$B = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Theorem

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible if and only if it is onto.

Theorem: A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is invertible iff it is one-to-one & onto

Proof: We need to construct an $S: \mathbb{R}^n \to \mathbb{R}^n$ such that $S \circ T(\vec{x}) = \vec{x}$ for all $\vec{x}$.

If $\vec{y} \in \mathbb{R}^n$ then there exists an $\vec{x} \in \mathbb{R}^n$ such that $T(\vec{x}) = \vec{y}$ since T is onto. Moreover $\vec{x}$ is unique since T is one-to-one

So, I define $S(\vec{y}) = \vec{x}$.

And then I see that $S \circ T(\vec{x}) = S(T(\vec{x})) = S(\vec{y}) = \vec{x}$

$\Rightarrow$ S is the inverse of T.

Theorem

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible if and only if it is onto.

Recall a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ is onto if and only it is one-to-one, so one may also check that it is one-to-on to check that it is invertible.

$$T(\vec{x}) = 2\vec{x} \qquad S(\vec{y}) = \frac{\vec{y}}{2}$$

$$T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1+x_2\\x_2\end{pmatrix} \qquad S\begin{pmatrix}y_1\\y_2\end{pmatrix} = \begin{pmatrix}y_1-y_2\\y_2\end{pmatrix} \qquad \begin{pmatrix}x_1\\x_2\end{pmatrix}$$

$$S\circ T\begin{pmatrix}x_1\\x_2\end{pmatrix} = S\left(T\begin{pmatrix}x_1\\x_2\end{pmatrix}\right) = S\begin{pmatrix}\overset{y_1}{x_1+x_2}\\ \underset{y_2}{x_2}\end{pmatrix} = \begin{pmatrix}y_1-y_2\\y_2\end{pmatrix} = \begin{pmatrix}x_1+x_2-x_2\\x_2\end{pmatrix}$$

Properties of Inverses

Theorem

1. *If T is invertible then T^{-1} is also invertible with $(T^{-1})^{-1} = T$*
2. *If T is one-to-one then T^{-1} exists and is also one-to-one*
3. *If T is onto then T^{-1} exists and is also onto*
4. *If T and S are invertible that $T \circ S$ is also invertible and $(T \circ S)^{-1} = S^{-1} \circ T^{-1}$*

$$\Updownarrow$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

Compositions and Matrix Multiplication

Theorem

Let A be an $m \times n$ matrix, B an $n \times k$ matrix and T_A, T_B the corresponding linear transformations. Then

$$T_A \circ T_B = T_{AB}$$

$T_A(\vec{x}) = A\vec{x}$

That is, $(T_A \circ T_B)(\vec{x}) = AB\vec{x}$ *for all* $\vec{x} \in \mathbb{R}^k$.

$$(T_A \circ T_B)(\vec{x}) = T_A(T_B(\vec{x})) = T_A(B\vec{x}) = AB\vec{x}$$

$$= T_{AB}(\vec{x})$$

Inverses and Matrix Inverses

Theorem

Let T be a linear transformation and let A be its standard matrix. Then T is invertible (as a transformation) if and only if A is invertible (as a matrix). Moreover,

$$T^{-1} = (T_A)^{-1} = T_{A^{-1}}$$

$$T_A \circ T_{A^{-1}} = T_{AA^{-1}} = T_{I_n} \iff T_{A^{-1}} = (T_A)^{-1}$$

Geometric Interpretation in $\mathbb{R}^2$

Recall: we said at the beginning that linear operators "preserves linear structure"...

Geometric Interpretation in $\mathbb{R}^2$

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Theorem

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an invertible linear operator. Then:

1. *The image of a line is a line*
2. *The image of a line passes through the origin if and only if the original line passes through the origin*
3. *The images of two lines are parallel if and only the original lines are parallel*

Geometric Interpretation in $\mathbb{R}^2$ continued

Theorem

4. *The images of three points lie on a line if and only if the original points lie on a line*

5. *The image of the line segment joining two points is the line segment joining the images of those points*