SF 1684 Algebra and Geometry
Lecture 10

Patrick Meisner

KTH Royal Institute of Technology

Topics for Today

1. Linear Transformations
2. Eigenvalues and Eigenvectors
3. Orthogonal Transformations

Linear Transformation

Definition

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ s called a **linear transformation** (or **linear map**) if for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$

1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$
2. $T(c\vec{x}) = cT(\vec{x})$

Definition

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ s called a **linear transformation** (or **linear map**) if for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$

1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$
2. $T(c\vec{x}) = cT(\vec{x})$

In general we can define linear transformation between any two vector space V and W in the same way.

Linear Transformation

Definition

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ s called a **linear transformation** (or **linear map**) if for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$

1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$
2. $T(c\vec{x}) = cT(\vec{x})$

In general we can define linear transformation between any two vector space V and W in the same way. Then we can think of linear transformation as functions that "preserving the linear structure of V in W".

Some Linear Transformations

1. $$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 5y \\ 2x - 3y \\ y \end{bmatrix}$$

Some Linear Transformations

1. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+5y \\ 2x-3y \\ y \end{bmatrix}$
2. Rotating each vector in $\mathbb{R}^2$ by $\pi/2$
3. Reflecting each vector in $\mathbb{R}^2$ in the line $y = x$
4. Projecting the vectors onto the x-axis
5. "Strecthing" by a factor of 2 in the x-direction

Four Basic Linear Transformations

Linear transformations come in four basic categories:

Four Basic Linear Transformations

Linear transformations come in four basic categories:

1. Rotation about the origin

Four Basic Linear Transformations

Linear transformations come in four basic categories:

1. Rotation about the origin
2. Reflection about a line

Four Basic Linear Transformations

Linear transformations come in four basic categories:

1. Rotation about the origin
2. Reflection about a line
3. Projection onto a line

Four Basic Linear Transformations

Linear transformations come in four basic categories:

1. Rotation about the origin
2. Reflection about a line
3. Projection onto a line
4. Stretching in the direction of a line

Four Basic Linear Transformations

Linear transformations come in four basic categories:

1. Rotation about the origin
2. Reflection about a line
3. Projection onto a line
4. Stretching in the direction of a line

Fact

All linear transformations can be broken up into components coming from these four basic categories.

Properties of Linear Tansformations

Theorem

Let T be any linear transformation. Then

1. $T(\vec{0}) = \vec{0}$
2. $T(-\vec{v}) = -T(\vec{v})$
3. $T(\vec{u} - \vec{v}) = T(\vec{u} - \vec{v})$

Matrices as Linear Transformations

Any $m \times n$ matrix, A, can define a linear transformation, T, from $\mathbb{R}^n$ to $\mathbb{R}^m$ by setting

$$T(\vec{x}) = A\vec{x}$$

Matrices as Linear Transformations

Any $m \times n$ matrix, A, can define a linear transformation, T, from $\mathbb{R}^n$ to $\mathbb{R}^m$ by setting

$$T(\vec{x}) = A\vec{x}$$

We will typically denote this transformation as T_A.

Matrices as Linear Transformations

Any $m \times n$ matrix, A, can define a linear transformation, T, from $\mathbb{R}^n$ to $\mathbb{R}^m$ by setting

$$T(\vec{x}) = A\vec{x}$$

We will typically denote this transformation as T_A.

For example: let

$$A = \begin{pmatrix} 1 & 5 \\ 2 & -3 \\ 0 & 1 \end{pmatrix}$$

Matrices as Linear Transformations

Any $m \times n$ matrix, A, can define a linear transformation, T, from $\mathbb{R}^n$ to $\mathbb{R}^m$ by setting

$$T(\vec{x}) = A\vec{x}$$

We will typically denote this transformation as T_A.

For example: let

$$A = \begin{pmatrix} 1 & 5 \\ 2 & -3 \\ 0 & 1 \end{pmatrix}$$

then the linear transformation we get from A will be

$$T_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

Matrices as Linear Transformations

Any $m \times n$ matrix, A, can define a linear transformation, T, from $\mathbb{R}^n$ to $\mathbb{R}^m$ by setting

$$T(\vec{x}) = A\vec{x}$$

We will typically denote this transformation as T_A.

For example: let

$$A = \begin{pmatrix} 1 & 5 \\ 2 & -3 \\ 0 & 1 \end{pmatrix}$$

then the linear transformation we get from A will be

$$T_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{pmatrix} 1 & 5 \\ 2 & -3 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrices as Linear Transformations

Any $m \times n$ matrix, A, can define a linear transformation, T, from $\mathbb{R}^n$ to $\mathbb{R}^m$ by setting

$$T(\vec{x}) = A\vec{x}$$

We will typically denote this transformation as T_A.

For example: let

$$A = \begin{pmatrix} 1 & 5 \\ 2 & -3 \\ 0 & 1 \end{pmatrix}$$

then the linear transformation we get from A will be

$$T_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{pmatrix} 1 & 5 \\ 2 & -3 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 5y \\ 2x - 3y \\ y \end{bmatrix}$$

Matrices as Linear Transformations

Any $m \times n$ matrix, A, can define a linear transformation, T, from $\mathbb{R}^n$ to $\mathbb{R}^m$ by setting

$$T(\vec{x}) = A\vec{x}$$

We will typically denote this transformation as T_A.

For example: let

$$A = \begin{pmatrix} 1 & 5 \\ 2 & -3 \\ 0 & 1 \end{pmatrix}$$

then the linear transformation we get from A will be

$$T_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{pmatrix} 1 & 5 \\ 2 & -3 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 5y \\ 2x - 3y \\ y \end{bmatrix}$$

Notice: this is the same linear transformation as the example on slide 4.

Matrices as Linear Transformations

Any $m \times n$ matrix, A, can define a linear transformation, T, from $\mathbb{R}^n$ to $\mathbb{R}^m$ by setting

$$T(\vec{x}) = A\vec{x}$$

We will typically denote this transformation as T_A.

For example: let

$$A = \begin{pmatrix} 1 & 5 \\ 2 & -3 \\ 0 & 1 \end{pmatrix}$$

then the linear transformation we get from A will be

$$T_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{pmatrix} 1 & 5 \\ 2 & -3 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 5y \\ 2x - 3y \\ y \end{bmatrix}$$

Notice: this is the same linear transformation as the example on slide 4.
Can we always find a matrix that defines the linear transformation?

Linear Transformations as Matrices

Theorem

Let T be any linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$.

Linear Transformations as Matrices

Theorem

Let T be any linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$. Define the matrix

$$A = \begin{pmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \ldots & T(\vec{e}_m) \end{pmatrix}$$

Linear Transformations as Matrices

Theorem

Let T be any linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$. Define the matrix

$$A = \begin{pmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \dots & T(\vec{e}_m) \end{pmatrix}$$

Then for all $\vec{x} \in \mathbb{R}^m$,

$$T(\vec{x}) = A\vec{x} \qquad (\text{or } T = T_A)$$

This matrix A is often called the **standard matrix of** T.

Exercise

Find the matrices that correspond to the linear transformations

1. Rotating each vector in $\mathbb{R}^2$ by $\pi/2$
2. Reflecting each vector in $\mathbb{R}^2$ in the line $y = x$
3. Projecting the vectors onto the x-axis
4. Stretching by a factor of 2 in the x-direction

More Work Space

Exercise

Find the matrix that corresponds to the linear transformation of rotating each vector in $\mathbb{R}^2$ by an angle θ.

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

Simplest Linear Transformations

Similar to how diagonal matrices are the simplest and easiest matrices, the transformation that correspond to diagonal matrices are the simplest and easiest as well.

Simplest Linear Transformations

Similar to how diagonal matrices are the simplest and easiest matrices, the transformation that correspond to diagonal matrices are the simplest and easiest as well. Let

$$D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$$

Simplest Linear Transformations

Similar to how diagonal matrices are the simplest and easiest matrices, the transformation that correspond to diagonal matrices are the simplest and easiest as well. Let

$$D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$$

Then the linear transformation that corresponds to D would be

$$T_D\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right)$$

Simplest Linear Transformations

Similar to how diagonal matrices are the simplest and easiest matrices, the transformation that correspond to diagonal matrices are the simplest and easiest as well. Let

$$D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$$

Then the linear transformation that corresponds to D would be

$$T_D\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Simplest Linear Transformations

Similar to how diagonal matrices are the simplest and easiest matrices, the transformation that correspond to diagonal matrices are the simplest and easiest as well. Let

$$D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$$

Then the linear transformation that corresponds to D would be

$$T_D\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 x_1 \\ d_2 x_2 \\ \vdots \\ d_n x_n \end{bmatrix}$$

Simplest Linear Transformations

Similar to how diagonal matrices are the simplest and easiest matrices, the transformation that correspond to diagonal matrices are the simplest and easiest as well. Let

$$D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$$

Then the linear transformation that corresponds to D would be

$$T_D\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \underline{d_1} x_1 \\ \underline{d_2} x_2 \\ \vdots \\ \underline{d_n} x_n \end{bmatrix}$$

In particular, we see that $\underline{T_{I_n}(\vec{x}) = \vec{x}}$ for all $\vec{x} \in \mathbb{R}^n$ $\quad I_n, \quad d_i = 1$

Simplest Action of a Linear Transformation

In particular, we note that

$$T_D(\vec{e}_i)$$

Simplest Action of a Linear Transformation

In particular, we note that

$$T_D(\vec{e}_i) = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

In particular, we note that

ith position

$$T_D(\vec{e}_i) = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots\, d_i & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ d_i \\ \vdots \\ 0 \end{bmatrix} = d_i \vec{e}_i$$

Simplest Action of a Linear Transformation

In particular, we note that

$$T_D(\vec{e_i}) = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ d_i \\ \vdots \\ 0 \end{bmatrix} = d_i \vec{e_i}$$

That is, we see that T_D acts on $\vec{e_i}$ in the simplest way it can:

Simplest Action of a Linear Transformation

In particular, we note that

$$T_D(\vec{e}_i) = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ d_i \\ \vdots \\ 0 \end{bmatrix} = d_i \vec{e}_i$$

That is, we see that T_D acts on $\vec{e}_i$ in the simplest way it can: by just multiplying by d_i.

Simplest Action of a Linear Transformation

In particular, we note that

$$T_D(\vec{e}_i) = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ d_i \\ \vdots \\ 0 \end{bmatrix} = d_i \vec{e}_i$$

That is, we see that T_D acts on $\vec{e}_i$ in the simplest way it can: by just multiplying by d_i.

Can this be extended too other matrices?

Eigenvalues and Eigenvectors

Definition

For any $n \times n$ matrix, A, we define λ to be an **eigenvalue** of A if there exists an non-zero vector $\vec{v}$ such that

$$A\vec{v} = \lambda\vec{v}.$$

Eigenvalues and Eigenvectors

Definition

For any $n \times n$ matrix, A, we define λ to be an **eigenvalue** of A if there exists an non-zero vector $\vec{v}$ such that $\lambda \in \mathbb{R}$

$$A\vec{v} = \lambda\vec{v}.$$

Metrix scalar

Moreover, we call such an $\vec{v}$ an **eigenvector** of A with eigenvalue λ.

not assuming true for all $\vec{v}$

just one $\vec{v}$.

Eigenvalues and Eigenvectors

Definition

For any $n \times n$ matrix, A, we define λ to be an **eigenvalue** of A if there exists an non-zero vector $\vec{v}$ such that

$$A\vec{v} = \lambda\vec{v}.$$

Moreover, we call such an $\vec{v}$ an **eigenvector** of A with eigenvalue λ.

Example:

$$\begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Eigenvalues and Eigenvectors

Definition

For any $n \times n$ matrix, A, we define λ to be an **eigenvalue** of A if there exists an non-zero vector $\vec{v}$ such that

$$A\vec{v} = \lambda\vec{v}.$$

Moreover, we call such an $\vec{v}$ an **eigenvector** of A with eigenvalue λ.

Example:

$$\begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Eigenvalues and Eigenvectors

Definition

For any $n \times n$ matrix, A, we define λ to be an **eigenvalue** of A if there exists an non-zero vector $\vec{v}$ such that

$$A\vec{v} = \lambda\vec{v}.$$

Moreover, we call such an $\vec{v}$ an **eigenvector** of A with eigenvalue λ.

Example:

$$\begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Eigenvalues and Eigenvectors

Definition

For any $n \times n$ matrix, A, we define λ to be an **eigenvalue** of A if there exists an non-zero vector $\vec{v}$ such that

$$A\vec{v} = \lambda\vec{v}.$$

Moreover, we call such an $\vec{v}$ an **eigenvector** of A with eigenvalue λ.

Example:

$$\begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \underline{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

And so we say that 2 is an **eigenvalue** of $\begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix}$

Eigenvalues and Eigenvectors

Definition

For any $n \times n$ matrix, A, we define λ to be an **eigenvalue** of A if there exists an non-zero vector $\vec{v}$ such that

$$A\vec{o} = \lambda \vec{o}$$

$$A\vec{v} = \lambda \vec{v}.$$

Moreover, we call such an $\vec{v}$ an **eigenvector** of A with eigenvalue λ.

Example:

$$\begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

And so we say that 2 is an **eigenvalue** of $\begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix}$ with **eigenvector** $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$$\begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \end{pmatrix} \neq 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Geometric Interpretation of Eigenvalues and Eigenvectors

Recall we stated that linear transformation have 4 basic forms

1. Rotation
2. Reflection about a line
3. Projection onto a line
4. Stretching in the direction of a line

Geometric Interpretation of Eigenvalues and Eigenvectors

Recall we stated that linear transformation have 4 basic forms

1. Rotation
2. Reflection about a line
3. Projection onto a line
4. Stretching in the direction of a line

Therefore, if $\vec{v}$ is an eigenvector of A with eigenvalues λ, then we can think of the linear transformation T_A, that corresponds by A, has a component corresponding to *stretching by a factor of λ in the direction of $\vec{v}$*.

v is a eigenvector of A with eigenvalue λ

$T_A(v) = \lambda v$. v gets stretched by a factor of λ.

Condition for Eigenvalues 1

Theorem

Let A be an $n \times n$ matrix. Then λ is an eigenvalue of A if and only if the matrix

$$A - \lambda I_n$$

has a non-trivial homogeneous solution. Moreover, all non-trivial homogeneous solutions to $A - \lambda I_n$ will be eigenvectors of A with eigenvalue λ.

($\Rightarrow$) if λ is an eigenvalue then there exists $v \neq 0$ such that $Av = \lambda v \Rightarrow Av - \lambda v = 0 \Rightarrow Av - \lambda I_n v = 0$

$\Rightarrow (A - \lambda I_n) v = 0 \Rightarrow v$ is a non-trivial homo solution to $A - \lambda I_n$.

($\Leftarrow$) if v is a hom solution to $A - \lambda I_n \Rightarrow$

$(A - \lambda I_n) v = 0 \Longrightarrow Av = \lambda v$

Condition for Eigenvalues 2

Theorem

Let A be an $n \times n$ matrix. Then the following are equivalent

1. λ *is an eigenvalue of* A
2. $A - \lambda I_n$ *has a non-trivial homogeneous solution*
3. $A - \lambda I_n$ *is not invertible*
4. $\det(A - \lambda I_n) = 0$

We've seen that $A - \lambda I_n$ has non-trivial homo solution

$\iff$ $A - \lambda I_n$ not invertible

$\iff$ $\det(A - \lambda I_n) = 0$

$\rightarrow$ polynomial in λ of deg n & eigenvalues will be the roots of the polynomial.

Major Theorem

Theorem

Let A be an $n \times n$ matrix. The the following are equivalent

1. $A\vec{x} = \vec{b}$ *has a unique solution for every* $\vec{b}$
2. $A\vec{x} = 0$ *has a unique solution*
3. $rk(A) = n$
4. *The RREF of A is I_n*
5. *A is invertible*
6. *The columns of A are linearly independent*
7. *The row vectors of A are linearly independent*
8. $\det(A) \neq 0$

Major Theorem

Theorem

Let A be an $n \times n$ matrix. The the following are equivalent

1. $A\vec{x} = \vec{b}$ *has a unique solution for every* $\vec{b}$
2. $A\vec{x} = 0$ *has a unique solution*
3. $rk(A) = n$
4. *The RREF of A is I_n*
5. *A is invertible*
6. *The columns of A are linearly independent*
7. *The row vectors of A are linearly independent*
8. $\det(A) \neq 0$
9. 0 *is* **not** *an eigenvalue of A*

0 is an eigenval iff $0 = \det(A - 0I_n) = \det(A)$

Example

Find the eigenvalues of eigenvectors of

$$A = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix}$$

Find λ such that $\det(A - \lambda I_n) = 0$

$$A - \lambda I_n = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} -\lambda & 1 \\ -6 & 5-\lambda \end{pmatrix}$$

$$\det(A - \lambda I_n) = \det\begin{pmatrix} -\lambda & 1 \\ -6 & 5-\lambda \end{pmatrix} = (-\lambda)(5-\lambda) - 1 \times (-6)$$

$$= \lambda^2 - 5\lambda + 6 = 0$$

$$\Rightarrow (\lambda - 3)(\lambda - 2) = 0$$

$$\Rightarrow \quad \lambda = 3 \quad \& \quad \lambda = 2$$

More Work Space

previous example we saw $\begin{pmatrix}0&1\\-6&5\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = 2\begin{pmatrix}1\\2\end{pmatrix}$

so $\begin{pmatrix}1\\2\end{pmatrix}$ is eigenvector with eigenvalue 2.

$\lambda=3$: v is an eigenvector if it is a homogeneous solution to $A-3I_n$

$$A-3I_n = \begin{pmatrix}0&1\\-6&5\end{pmatrix} - \begin{pmatrix}3&0\\0&3\end{pmatrix} = \begin{pmatrix}-3&1\\-6&2\end{pmatrix} \xrightarrow{R_2-2R_1} \begin{pmatrix}-3&1\\0&0\end{pmatrix} \xrightarrow{-\frac{1}{3}R_1} \begin{pmatrix}1&-\frac{1}{3}\\0&0\end{pmatrix}$$

$$\begin{pmatrix}1&-\frac{1}{3}\\0&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x-\frac{1}{3}y\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \begin{array}{l} x = \frac{1}{3}y \\ y = t \end{array}$$

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\frac{1}{3}t\\t\end{pmatrix} = t\begin{pmatrix}\frac{1}{3}\\1\end{pmatrix}$$

$\begin{pmatrix}1/3\\1\end{pmatrix}t$ is an eigenvector with eigenvalue 3 for all t.

set $t=3$: $\begin{pmatrix}1\\3\end{pmatrix}$ is an eigenvector with eigenvalue 3.

Orthogonal Transformations

We have seen that eigenvalues can describe the component of a linear operation that corresponds to stretching in a direction.

Orthogonal Transformations

We have seen that eigenvalues can describe the component of a linear operation that corresponds to stretching in a direction. But what about the other components?

Orthogonal Transformations

We have seen that eigenvalues can describe the component of a linear operation that corresponds to stretching in a direction. But what about the other components?

Definition

We say a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is **orthogonal** if

$$\|T(\vec{x})\| = \|\vec{x}\|$$

for all $\vec{x} \in \mathbb{R}^n$.

Orthogonal Transformations

We have seen that eigenvalues can describe the component of a linear operation that corresponds to stretching in a direction. But what about the other components?

Definition

We say a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is **orthogonal** if

$$\|T(\vec{x})\| = \|\vec{x}\|$$

for all $\vec{x} \in \mathbb{R}^n$. We sometimes call this property norm preserving

Dot-Product Preserving

Theorem

A linear transformation, $T : \mathbb{R}^n \to \mathbb{R}^n$, is orthogonal if and only if $T(\vec{x}) \cdot T(\vec{y}) = \vec{x} \cdot \vec{y}$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$.

Hence, we sometimes call orthogonal transformations *dot-product preserving*.

If $T(\vec{x}) \cdot T(\vec{y}) = \vec{x} \cdot \vec{y}$ for all $\vec{x}, \vec{y}$

$T(\vec{x}) \cdot T(\vec{x}) = \vec{x} \cdot \vec{x} \implies \|T(\vec{x})\| = \|\vec{x}\|$

$\|T(\vec{x})\|^2 = \|x \cdot x\|^2$

Examples of Orthogonal Transformation

For any θ, the linear transformation given by $T(\vec{x}) = A\vec{x}$ is orthogonal

$$A := \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

Geometrically A corresponds rotating by an angle of θ and so does not change any lengths.

Exercise!
$$\left\| \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\| = \left\| \begin{pmatrix} (\cos\theta)x - (\sin\theta)y \\ (\sin\theta)x + (\cos\theta)y \end{pmatrix} \right\|$$

$$= \left[\left((\cos\theta)x - (\sin\theta)y \right)^2 + \left((\sin\theta)x + (\cos\theta)y \right)^2 \right]^{1/2} = \cdots = \sqrt{x^2 + y^2}$$

Orthogonal Matrices

Definition

We say a square matrix A is **orthogonal** if the linear transformation $T(\vec{x}) = A\vec{x}$ is orthogonal.

Theorem

The following statements are equivalent

1. *A is orthogonal*

Orthogonal Matrices

Definition

We say a square matrix A is **orthogonal** if the linear transformation $T(\vec{x}) = A\vec{x}$ is orthogonal.

Theorem

The following statements are equivalent

1. *A is orthogonal*
2. $||A\vec{x}|| = ||\vec{x}||$ *for all* $\vec{x} \in \mathbb{R}^n$

Orthogonal Matrices

Definition

We say a square matrix A is **orthogonal** if the linear transformation $T(\vec{x}) = A\vec{x}$ is orthogonal.

Theorem

The following statements are equivalent

1. *A is orthogonal*
2. $||A\vec{x}|| = ||\vec{x}||$ *for all* $\vec{x} \in \mathbb{R}^n$
3. $(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$ *for all* $\vec{x}, \vec{y} \in \mathbb{R}^n$

means for $T(x) = A\vec{x}$ to be orthogonal

Orthogonal Matrices

Definition

We say a square matrix A is **orthogonal** if the linear transformation $T(\vec{x}) = A\vec{x}$ is orthogonal.

Theorem

The following statements are equivalent

1. A *is orthogonal*
2. $||A\vec{x}|| = ||\vec{x}||$ *for all* $\vec{x} \in \mathbb{R}^n$
3. $(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$ *for all* $\vec{x}, \vec{y} \in \mathbb{R}^n$
4. $A^T A = I_n$

Orthogonal Matrices

Definition

We say a square matrix A is **orthogonal** if the linear transformation $T(\vec{x}) = A\vec{x}$ is orthogonal.

Theorem

The following statements are equivalent

1. *A is orthogonal*
2. $||A\vec{x}|| = ||\vec{x}||$ *for all* $\vec{x} \in \mathbb{R}^n$
3. $(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$ *for all* $\vec{x}, \vec{y} \in \mathbb{R}^n$
4. $A^T A = I_n$ by definition of A^{-1}
5. $A^T = A^{-1}$

Orthogonal Matrices

Definition

We say a square matrix A is **orthogonal** if the linear transformation $T(\vec{x}) = A\vec{x}$ is orthogonal.

Theorem

The following statements are equivalent

1. *A is orthogonal*
2. $||A\vec{x}|| = ||\vec{x}||$ *for all* $\vec{x} \in \mathbb{R}^n$
3. $(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$ *for all* $\vec{x}, \vec{y} \in \mathbb{R}^n$
4. $A^T A = I_n$
5. $A^T = A^{-1}$
6. *Any two column vectors of A are orthogonal and unit vectors*

Sketch of Proof

(3) $\iff$ (4) $\quad (Ax)\cdot(Ay) = x\cdot y$ for all x, y

$$(Ax)^T(Ay) = x^T y$$

$$x^T A^T A y = x^T I_n y \quad \text{for all } \underline{x, y}$$

needs more work $\implies$ $\quad A^T A = I_n$

(4) $\iff$ (6) $\quad A = \begin{pmatrix} c_1 & \cdots & c_n \end{pmatrix} \qquad A^T = \begin{pmatrix} c_1^T \\ \vdots \\ c_n^T \end{pmatrix}$

$$A^T A = \begin{pmatrix} c_1^T \\ \vdots \\ c_n^T \end{pmatrix} \begin{pmatrix} c_1 & \cdots & c_n \end{pmatrix} = \begin{pmatrix} c_i \cdot c_j \end{pmatrix}_{i,j} = I_n$$

if $i \neq j \implies c_i \cdot c_j = 0 \implies$ j^{th} column is orth to i^{th} column

if $i = j \implies c_i \cdot c_i = 1 \implies \|c_i\| = 1$ $\quad i^{th}$ column is unit.

Properties of Orthogonal Matrices 1

Theorem

If A is an orthogonal matrix that $\det(A) = 1$ *or* -1.

$A^T A = I_n$ so taking determinants

$\det(A^T A) = \det(I_n) = 1$

$\det(A^T)\det(A) = \det(A)\cdot\det(A) = \det(A)^2 = 1$

$\Rightarrow \det(A) = 1$ or -1

Properties of Orthogonal Matrices 2

Theorem

1. *The product of two orthogonal matrices is orthogonal*
2. *The inverse of an orthogonal matrix is orthogonal*
3. *The transpose of an orthogonal matrix is orthogonal*
4. *A is orthogonal if and only if it's row vectors are orthonormal*

A, B orthog $\Rightarrow$ AB ortho

A orthog $\Rightarrow$ A^{-1} ortho

A ortho $\Rightarrow$ A^{T} ortho

Exercise prove this

orthogonal & unit.

4 follows immediately from 3 & previous thm.