SF 1684 Algebra and Geometry
Lecture 9

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Topics for Today

1. Using Determinants to Solve Matrix Equations: Cramer's Rule
2. Geometric Interpretation of Determinants
3. Cross Products and Determinants

Solving Matrix Equations

We now know that if A is an invertible matrix then there is always a unique solution to

$$A\vec{x} = \vec{b}$$

for every $\vec{b}$, namely $A^{-1}\vec{b}$.

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We have an algorithm for finding the inverse but can we find a formula?

Adjoint of a Matrix

Recall, for a matrix A, we define the (i,j)-th cofactor, $C_{i,j}$ to be the signed determinant of the matrix obtained by removing i-th row and j-th column from A.

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Definition

For any matrix A, we define the **matrix of cofactors** of A to be

$$C = \begin{pmatrix} C_{1,1} & C_{1,2} & \dots & C_{1,n} \\ C_{2,1} & C_{2,2} & \dots & C_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n,1} & C_{n,2} & \dots & C_{n,n} \end{pmatrix}$$

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We define the **adjoint** of A (denoted $\mathrm{adj}(A)$) to be

$$\mathrm{adj}(A) = C^T$$

Formula for Inverse

det is a scalar

Theorem

If A is an invertible matrix then

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

Sketch of Proof.

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Sketch of Proof.

Use the definition of $\mathrm{adj}(A)$ to show that

$$\frac{1}{\det(A)} A \cdot \mathrm{adj}(A) = \frac{1}{\det(A)} \mathrm{adj}(A) \cdot A = I_n$$

$\square$

Example

Let

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}$$

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$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix} \qquad \begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

Then

$$C_{1,1} = 12 \qquad C_{1,2} = 6 = -(1 \times 0 - 2 \times 2)$$

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$$C_{1,1} = 12 \qquad C_{1,2} = 6 \qquad C_{1,3} = -16 \quad = 1\times -4 - 2\times 6$$

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Then

$$C_{1,1} = 12 \qquad C_{1,2} = 6 \qquad C_{1,3} = -16$$

$$C_{2,1} = 4 \qquad C_{2,2} = 2 \qquad C_{2,3} = 16$$

$$C_{3,1} = 12 \qquad C_{3,2} = -10 \qquad C_{3,3} = 16$$

Example

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$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}$$

Then

$$\begin{array}{ccc} C_{1,1} = 12 & C_{1,2} = 6 & C_{1,3} = -16 \\ C_{2,1} = 4 & C_{2,2} = 2 & C_{2,3} = 16 \\ C_{3,1} = 12 & C_{3,2} = -10 & C_{3,3} = 16 \end{array}$$

and so

$$\begin{aligned} \det(A) &= 3 * 12 + 2 * 6 + (-1) * (-16) \\ &= 1 * 4 + 6 * 2 + 3 * 16 \\ &= 2 * 12 + (-4) * (-10) + 0 * 16 \\ &= 2 * 6 + 6 * 2 + (-4) * (-10) \\ &= 64 \end{aligned}$$

1st row

2nd row

3rd row

2nd col

Example Continued

$$C = \begin{pmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{pmatrix}$$

Example Continued

$$C = \begin{pmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{pmatrix} \qquad \text{adj}(A) = C^T = \begin{pmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{pmatrix}$$

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Exercise

Check that

$$\det A \quad \frac{1}{64} \overbrace{\begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}}^{A} \overbrace{\begin{pmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{pmatrix}}^{\text{adj } A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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$$x_1 = \frac{\det(A_1)}{\det(A)} \qquad x_2 = \frac{\det(A_2)}{\det(A)} \qquad \dots \qquad x_n = \frac{\det(A_n)}{\det(A)}$$

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We see here that we can find a solution by only taking $n+1$ determinants!

Example

Solve the system

$$\begin{aligned} x_1 + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

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Example Continued

Computing, we find that

$$\det(A) = 44 \qquad \det(A_1) = -40 \qquad \det(A_2) = 72 \qquad \det(A_3) = 152$$

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Hence,

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Example Continued

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$$\det(A) = 44 \qquad \det(A_1) = -40 \qquad \det(A_2) = 72 \qquad \det(A_3) = 152$$

Hence,

imperative that A is invertible
since we are dividing by the det(A)

$$\vec{x} = \begin{bmatrix} \det(A_1)/\det(A) \\ \det(A_2)/\det(A) \\ \det(A_3)/\det(A) \end{bmatrix} = \begin{bmatrix} -40/44 \\ 72/44 \\ 152/44 \end{bmatrix} = \begin{bmatrix} -10/11 \\ 18/11 \\ 38/11 \end{bmatrix}$$

is the unique solution to the system of linear equations.

A is not invertible the RREF $(0 \cdots 0 \mid *)$

$* \neq 0$
$\Rightarrow$ solution

Example Continued

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Exercise

Prove Cramer's Rule. Hint: use the adjoint formula for the inverse.

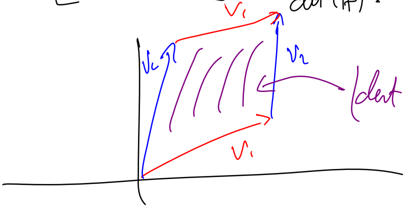
Geometric Interpretation of Determinants

Theorem

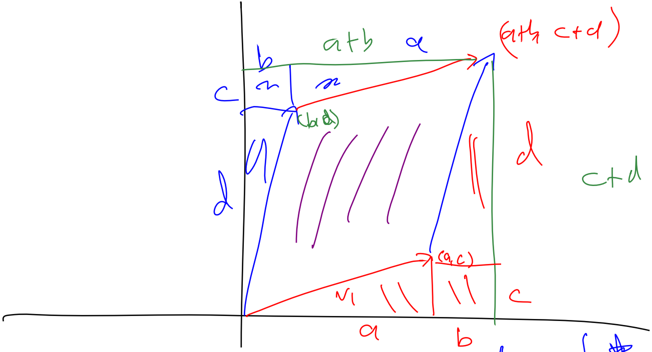
1. *If A is a 2×2 matrix, then $|\det(A)|$ represents the* **area** *of the parallelogram determined by the two column vectors of A.*
2. *If A is a 3×3 matrix, then $|\det(A)|$ represents the* **volume** *of the parallelepiped determined the by the three columns of A.*
3. *In general, $|\det(A)|$ can be thought of as an "n-dimensional volume" of the n-dimensional object determined by the columns of A.*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad v_1 = \begin{bmatrix} a \\ c \end{bmatrix} \qquad v_2 = \begin{bmatrix} b \\ d \end{bmatrix}$$

$\det(A) = ad - bc =$ area of the paral.. determined by $v_1 \& v_2$



$|\det A| = |ad - bc|$



Area of # $= \frac{bd}{2} + bc + \frac{ac}{2}$

Area of # $= (a+b)(c+d)$

Area of # $= \frac{ac}{2} + \frac{bd}{2} + bc$

Area of # $= (a+b)(c+d)$
$- \left(\frac{ac}{2} + \frac{bd}{2} + bc\right)$
$- \left(\frac{bd}{2} + bc + \frac{ac}{2}\right)$
$= \dots = ad - bc = \det(A)$

Geometric Interpretation of Determinant Zero

Question

What is the geometric interpretation of a matrix having determinant 0?

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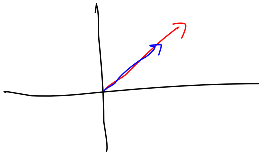
Answer: the n-dimensional objects determined by the columns of A actually lives in an $n-1$-dimensional space and thus has 0 "n-dimensional volume". the columns of a 3x3 matrix with det 0 form a 2-dimensional plane instead of a 3-dim parallelopiped.

That is, the determinant of a 2×2 matrix is 0 if and only if its columns are proportional if and only if the "parallelogram" determined by its columns is a line and hence has 0 area.

$v_1 \quad v_2 \longrightarrow$ area 0 if they determine a line



Brief Aside to Linear Functions

We can think of an $m \times n$ matrix as a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. Indeed:

$$\begin{aligned} \mathbb{R}^n &\to \mathbb{R}^m \\ \vec{x} &\mapsto A\vec{x} \end{aligned}$$

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Moreover, A would take a shape of "perimeter" L and map it to a shape of "perimeter" $|\mathrm{Tr}(A)|L$.

Cross Product as Determinant

Recall, for vectors in $\mathbb{R}^3$, we define the cross product

looks like a 2x2 determinant

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \det A = ad - bc$

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This definition should now evoke notions of determinants. Indeed:

$$\vec{u} \times \vec{v} = \left(\det \begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix}, -\det \begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix}, \det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \right)$$

$u_2 v_3 - u_3 v_2$ $\quad$ $-(u_1 v_3 - u_3 v_1)$

A good way to remember the formula for the cross product is as the determinant of a "formal matrix":

$$\vec{u} \times \vec{v} = \det \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

$\vec{e}_1, \vec{e}_2, \vec{e}_3$ standard vectors

$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ...

$$\det\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} = \vec{e}_1 \det\begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix} - \vec{e}_2 \det\begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix} + \vec{e}_3 \det\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}$$

$$= \left(\det\begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix}, -\det\begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix}, \det\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \right)$$

$$= \vec{u} \times \vec{v}$$

Example

Use the “formal matrix” to calculate

$$\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \times \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ -6 \end{bmatrix}$$

$$\det\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{pmatrix} = \vec{e}_1 \det\begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix} - \vec{e}_2 \det\begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix} + \vec{e}_3 \det\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$$

$$= \vec{e}_1 (2\times 1 - 0\times -2) - \vec{e}_2 \left(1\times 1 - 3\times -2\right) + \vec{e}_3 \left(1\times 0 - 2\times 3\right)$$

$$= 2\vec{e}_1 - 7\vec{e}_2 - 6\vec{e}_3 = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 7\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 6\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{pmatrix} 2 \\ -7 \\ -6 \end{pmatrix}$$

Cross Product Theorem

We can now use this notation of cross product to prove many properties of the cross product easily

Theorem

Let $\vec{u}$ and $\vec{v}$ be two vectors in $\mathbb{R}^3$ and c and real number. Then

1. $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
2. $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$
3. $\vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0}$
4. $\vec{u} \times \vec{u} = \vec{0}$

(1) $$\vec{u} \times \vec{v} = \det\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

$$\vec{v} \times \vec{u} = \det\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{pmatrix}$$

going between these matrices is a row swap! And we have seen that a row swap changes the sign of your determinant.

More Work Space

(2) $(c\vec{u})\times\vec{v} = \det\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ cu_1 & cu_2 & cu_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$ ← This matrix is obtained by multiplying a row by a constant! We have seen that this changes the determinant by a the multiple.

$c\cdot(\vec{u}\times\vec{v}) = c\cdot\det\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$

(3) $\vec{u}\times\vec{0} = \det\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ u_1 & u_2 & u_3 \\ 0 & 0 & 0 \end{pmatrix} = 0$ because one row is 0.

(4) $\vec{u}\times\vec{u} = \det\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \end{pmatrix} = 0$ because two rows are proportional

The Standard Basis Vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$

Oftein in $\mathbb{R}^3$, we write we write $\vec{e}_1 = \mathbf{i}$, $\vec{e}_2 = \mathbf{j}$ and $\vec{e}_3 = \mathbf{k}$.

The Standard Basis Vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$

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$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\left[\; e_1 \times e_2 = e_3 \qquad e_2 \times e_3 = e_1 \qquad e_3 \times e_1 = e_2 \;\right]$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

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– straight calculation

can also see this through straight calculation.

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

– Swapping two vectors changes sign

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$$\mathbf{i} \times \mathbf{i} = \vec{0} \qquad \mathbf{j} \times \mathbf{j} = \vec{0} \qquad \mathbf{k} \times \mathbf{k} = \vec{0}$$

- From this any vector crossed with itself is 0.

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Now, any vector in $\mathbb{R}^3$ can be written as linear combination of $\vec{e}_1, \vec{e}_2$ and $\vec{e}_3$ and hence of $\mathbf{i}$,$\mathbf{j}$ and $\mathbf{k}$. Therefore, to cross multiply two vectors it is enough to "expand the product":

$$\vec{u} \times \vec{v} = (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$$

Example

"Expand the product" to calculate

$$\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \times \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ -2 \end{bmatrix}$$

$$(i + 2j - 2k) \times (3i + 0j + 1k) = \cancel{3i \times i}^{\,0} + \cancel{0 i \times j}^{\,0} + 1\, i \times k$$
$$+ 2j \times i + \cancel{0 j \times j}^{\,0} + 2j \times k$$
$$- 6k \times i + \cancel{0 k \times j}^{\,0} - \cancel{2k \times k}^{\,0}$$

$$= -j - 2k + 2i - 6j$$

$$= 2i - 7j - 2k = \begin{bmatrix} 2 \\ -7 \\ -2 \end{bmatrix}$$

Big Caution

Note that:

$$\mathbf{i} \times (\mathbf{j} \times \mathbf{j}) = \mathbf{i} \times \vec{0} = \vec{0}$$

while

$$(\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = \mathbf{k} \times \mathbf{j} = -\mathbf{i} \neq \vec{0}$$

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Hence, it is not true in general that

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Hence, it is not true in general that

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The cross product is NOT associative. So writing something like

$$u \times (v \times w) \quad \stackrel{?}{=} \quad \vec{u} \times \vec{v} \times \vec{w} \quad \stackrel{?}{=} \quad (u \times v) \times w$$

does NOT make sense as it depends on the order you are (cross) multiplying them in.

Dot Product with **i**,**j**,**k**

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$$\mathbf{i}\cdot\mathbf{i}=1 \qquad \mathbf{j}\cdot\mathbf{j}=1 \qquad \mathbf{k}\cdot\mathbf{k}=1$$

$$\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\0\\0\end{bmatrix}$$

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Then we can define dot product of two vectors by "expanding the product"

$$\vec{u}\cdot\vec{v}=(u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k})\cdot(v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k})$$

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$$i^T i = \mathbf{i}\cdot\mathbf{i} = 1 \qquad \mathbf{j}\cdot\mathbf{j} = 1 \qquad \mathbf{k}\cdot\mathbf{k} = 1$$

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{i} = \mathbf{i}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{j} = 0$$

Then we can define dot product of two vectors by "expanding the product"

$$\vec{u}\cdot\vec{v} = (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k})\cdot(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$$

Exercise

Use the "expand the product" idea to prove

1. $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v} + \vec{u}\times\vec{w}$
2. $(\vec{u}+\vec{v})\times\vec{w} = \vec{u}\times\vec{w} + \vec{v}\times\vec{w}$
3. $\vec{u}\cdot(\vec{u}\times\vec{v}) = 0$ (i.e. $\vec{u}$ is orthogonal to $\vec{u}\times\vec{v}$)
4. $\vec{v}\cdot(\vec{u}\times\vec{v}) = 0$ (i.e. $\vec{v}$ is orthogonal to $\vec{u}\times\vec{v}$)

(2) is not implied by (1)

Cross Product as an Area

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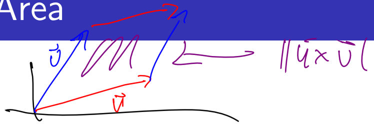
Theorem

Let $\vec{u}$ and $\vec{v}$ be non-zero vectors in $\mathbb{R}^3$ and let θ be the angle between them. Then

1. $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \, \|\vec{v}\| \sin(\theta)$

$|u \cdot v| = \|u\| \cdot \|v\| \cos\theta$

Cross Product as an Area



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1. $\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\,\|\vec{v}\|\sin(\theta)$
2. $\|\vec{u}\times\vec{v}\|$ *is the area of the parallelogram that has $\vec{u}$ and $\vec{v}$ as adjacent sides.*

proof 2) formula for area of parallelogram

Area of parallelogram = (side length 1) × (side length 2) × sin (angle)

Proof

More Work Space