# SF 1684 Algebra and Geometry Lecture 7

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KTH Royal Institute of Technology

- Easily Invertible Matrices
- In Functions on Matrices: Transpose and Trace
- Subspaces
- Linear Dependence

# Easily Invertible Matrices: Diagonal Matrices

Let

$$D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$$

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$$D^{-1} = \begin{pmatrix} \frac{1}{d_1} & 0 & \dots & 0\\ 0 & \frac{1}{d_2} & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \frac{1}{d_p} \end{pmatrix}$$

## Easily Ivertible Matrices: $2 \times 2$ Matrix

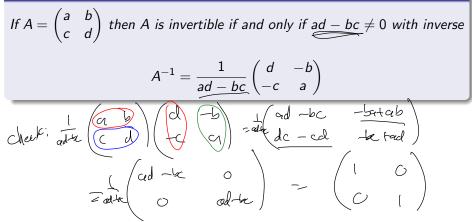
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### Easily Ivertible Matrices: $2 \times 2$ Matrix

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However, in the  $2 \times 2$  case, there is a simple formula:

#### Theorem



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Example:

$$A := \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \implies A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

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**5** 
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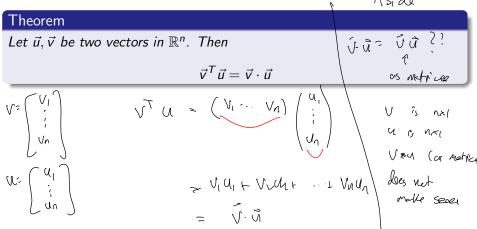
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### Definition

For a square matrix  $A = (a_{i,j})$ , we define the **trace** of the matrix as the sum of its diagonal entries:

$$\mathsf{Tr}(A) = \mathsf{a}_{1,1} + \mathsf{a}_{2,2} + \cdots + \mathsf{a}_{n,n}$$

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Example:

$$\operatorname{Tr}\left(\begin{pmatrix} 3 & 6 & -1000 & 2\\ 9001 & 1 & 44 & 54\\ 0 & 789134 & 1 & 98\\ -578 & 913 & 1 & 2 \end{pmatrix}\right) = 3 + 1 + 1 + 2 = 7$$

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Let A and B be  $n \times n$  square matrices and c any real number

Note: While it is almost never true that AB = BA, it happens that it is always true that Tr(AB) = Tr(BA).

### Dot Product as a Trace

Recall that if  $\vec{u}, \vec{v}$  are vectors in  $\mathbb{R}^n$ , then we can think of them as  $n \times 1$  matrix and  $\vec{v}^T$  as a  $1 \times n$  matrix.

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#### Theorem

Let  $\vec{u}, \vec{v}$  be two matrices in  $\mathbb{R}^n$ . Then  $\vec{v}\vec{u}^T$  is a square  $n \times \underline{n}$  matrix and  $Tr(\vec{v}\vec{u}^T) = \vec{v}\cdot\vec{u}$  $\begin{aligned} u_{\tau} \begin{pmatrix} u_{i} \\ i \\ u_{n} \end{pmatrix} & T_{r} (V \ u^{T}) = U_{i} V_{i} + U_{2} V_{1} + \dots + U_{n} V_{n} \\ &= \overline{U} \cdot \overline{V} = \overline{V} \cdot \overline{v} \end{aligned}$ 

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#### Exercise

If W is a subspace of  $\mathbb{R}^n$ , then show that  $\vec{0} \in W$ .

# Smallest Subspace

## Definition

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#### Exercise

Show that the zero subspace actually is a subspace of  $\mathbb{R}^n$ .

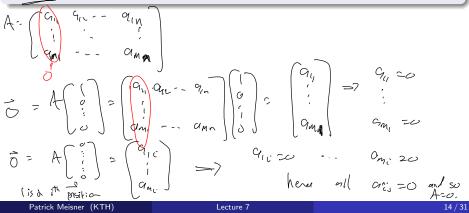
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chade: non-entry. 
$$\vec{0}$$
 is clurarys q homogenery solution.  
check: scolar mult:  $\vec{x}$  homo, sil cell  $A(c\vec{x}) = c(A\vec{x}) = c\vec{0} = \vec{3}$   
so  $c\vec{3}$  is homo. solution.  
dreek: addition:  $\vec{x}, \vec{y}$  and homo sol.  $A(\vec{z}+\vec{y}) = A\vec{x} + A\vec{y} = \vec{3} + \vec{3} = \vec{3}$   
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If A and B are matrices with n columns, then  $A\vec{x} = B\vec{x}$  for all  $\vec{x} \in \mathbb{R}^n$  if and only if A = B.

Let  $\vec{v_1}, \vec{v_2}, \dots, \vec{v_k}$  be vectors in  $\mathbb{R}^n$ , then the set of all linear combinations of them

$$W = \{ \vec{x} = t_1 \vec{v}_1 + t_2 \vec{v}_2 + \dots + t_k \vec{v}_k : t_1, t_2, \dots, t_k \in \mathbb{R} \}$$

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check: Wis non-crupty: Obvinus as chara and 6... the GIR check: Scalar nultiplications xew cell C-x: c (b.VI+b.VI+-> b.v.Vie) = (b)VI+...+(b)Vie GW

$$\frac{d}{x} + \frac{v}{y} = \left(\frac{1}{v_1} + \frac{v_1}{v_1} + \frac{v_2}{v_2}\right) + \left(\frac{1}{v_1} + \frac{v_2}{v_1} + \frac{v_2}{v_2}\right) + \left(\frac{1}{v_1} + \frac{v_2}{v_1}\right) + \left(\frac{v_2}{v_1} + \frac{v_2}{v_2}\right) + \left(\frac{v_2}{v_1} + \frac{v_2}{v_2}\right) + \left(\frac{v_2}{v_1} + \frac{v_2}{v_1}\right) + \left(\frac{v_2}{v_1} + \frac{v_2}{v_2}\right) + \left(\frac{v_2}{v_1} + \frac{v_2}{v_1}\right) + \left(\frac{v_2}{v_1} + \frac{v_2}{v_1}\right$$

The example in the previous slide is called the **span** of the vectors  $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$ . We often denote it

$$W = \{\vec{x} = t_1 \vec{v}_1 + t_2 \vec{v}_2 + \dots + t_k \vec{v}_k : t_1, t_2, \dots, t_k \in \mathbb{R}\}$$
  
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### Recall we have the **standard normal vectors** of $\mathbb{R}^n$

$$\vec{e_1} = \begin{bmatrix} 1\\0\\\vdots\\0 \end{bmatrix} \quad \vec{e_2} = \begin{bmatrix} 0\\1\\\vdots\\0 \end{bmatrix} \quad \dots \quad \vec{e_n} = \begin{bmatrix} 0\\0\\\vdots\\1 \end{bmatrix}$$

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So  $\vec{e_1}, \ldots, \vec{e_n}$  is a spanning set of vectors for  $\mathbb{R}^n$ 

Find the subspace of homogeneous solutions of

$${f A}=egin{pmatrix} 1&3&-2&0&2&0\ 2&6&-5&-2&4&-3\ 0&0&5&10&0&15\ 2&6&0&8&4&18 \end{pmatrix}$$

and write it as a span of vectors.

Find the subspace of homogeneous solutions of

$$\mathsf{A} = \begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{pmatrix}$$

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We know that set of homogeneous solutions will all those  $\vec{x}$  that satisfy  $A\vec{x} = \vec{0}$ .

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$$\begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{pmatrix} \implies \begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 0 \\ 0 & 0 & 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

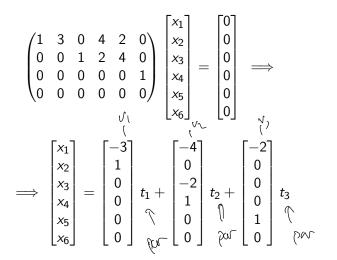
## Exercise Continued

Hence, we need to find the set of  $\vec{x}$  such that

$$\begin{pmatrix} \chi_{1} & \chi_{4} & \chi_{5} \\ 1 & 3 & 0 & 4 & 2 & 0 \\ 0 & 0 & 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} \chi_{1} \\ \chi_{2} \\ \chi_{3} \\ \chi_{4} \\ \chi_{5} \\ \chi_{6} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

## Exercise Continued

Hence, we need to find the set of  $\vec{x}$  such that



Therefore, we can conclude that the subspace of homogeneous solutions to A is

$$\begin{cases} \begin{bmatrix} -3\\1\\0\\0\\0\\0 \end{bmatrix} t_{1} + \begin{bmatrix} -4\\0\\-2\\1\\0\\0\\0 \end{bmatrix} t_{2} + \begin{bmatrix} -2\\0\\0\\0\\1\\0\\1\\0 \end{bmatrix} t_{3} : t_{1}, t_{2}, t_{3} \in \mathbb{R} \end{cases}$$
$$= \operatorname{span} \begin{cases} \begin{bmatrix} -3\\1\\0\\0\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} -4\\0\\-2\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix} -4\\0\\0\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} -2\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix} \end{cases}$$

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so that  $\vec{e_1}, \ldots, \vec{e_n}$  is a spanning set of vectors. However, we can find many different spanning sets of vectors for the same vector space

$$\mathbb{R}^2 = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$\mathbb{R}^n = \operatorname{span}\{\vec{e_1}, \ldots, \vec{e_n}\}$$

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exercise. chark this

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### Redundancy in Spanning Sets

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so any linear combination of all three of the vectors can easily be written as a linear combination of first two:

$$\begin{aligned} t_1 \begin{bmatrix} 1\\0 \end{bmatrix} + t_2 \begin{bmatrix} 1\\1 \end{bmatrix} + t_3 \begin{bmatrix} 2\\1 \end{bmatrix} &= t_1 \begin{bmatrix} 1\\0 \end{bmatrix} + t_2 \begin{bmatrix} 1\\1 \end{bmatrix} + t_3 \left( \begin{bmatrix} 1\\0 \end{bmatrix} + \begin{bmatrix} 1\\1 \end{bmatrix} \right) \\ &= (t_1 + t_3) \begin{bmatrix} 1\\0 \end{bmatrix} + (t_2 + t_3) \begin{bmatrix} 1\\1 \end{bmatrix} \end{aligned}$$

We then say that these three vectors are **linearly dependent**.

Patrick Meisner (KTH)

# Linearly Dependent and Independent

#### Definition

We say that a set of vectors  $\vec{v}_1, \ldots, \vec{v}_k$  are **linearly dependent** if there is a  $\vec{v}_i$  that can be written as a linear combination of the rest of the vectors:

$$\vec{v_i} = t_1 \vec{v_1} + t_2 \vec{v_2} + \dots + t_{i-1} \vec{v_{i-1}} + t_{i+1} \vec{v_{i+1}} + \dots + t_k \vec{v_k}$$

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If no such relationship exists we say the vectors are linearly independent.

#### Fact

Any set of vectors containing  $\vec{0}$  is linearly dependent.

$$V_{1}$$
...  $V_{e_{i}}$  could  $V_{i} = \overset{\circ}{O}$  Her

### Creating Linearly Independence out of Linearly Dependence

If  $\vec{v_1}, \ldots, \vec{v_k}$  is a linearly dependent set of vectors with  $\vec{v_i}$  being able to be written as a linear combination of the rest of the vectors, then we get

$$span\{\vec{v}_{1},...,\vec{v}_{k}\} = span\{\vec{v}_{1},...,\vec{v}_{i-1},\vec{v}_{i+1},...,\vec{v}_{k}\}$$

$$f_{i} \vee_{i} + f_{i} \vee_{i} + \cdots + f_{n} \vee_{n} = f_{i} \vee_{i} \vee_{i} + f_{i} \vee_{n} + f_{n} \vee_{n} \vee_{n} + f_{n} \vee_{n} \vee_{n} + f_{n} \vee_{n} \vee_{n} \vee_{n} + f_{n} \vee_{n} \vee_{n} \vee_{n} + f_{n} \vee_{n} \vee_{n}$$

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Hence, as long as the remaining vectors are still linearly dependent, then we can keep removing one vector without affecting the span of the vectors.

#### Theorem

For any set of vectors  $\vec{v}_1, \ldots, \vec{v}_k$ , we can find a subset of the vectors  $\vec{v}_{i_1}, \ldots, \vec{v}_{i_\ell}$  such that

$$span\{\vec{v}_1,\ldots,\vec{v}_k\}=span\{\vec{v}_{i_1},\ldots,\vec{v}_{i_\ell}\}$$

and  $\vec{v}_{i_1}, \ldots, \vec{v}_{i_{\ell}}$  is linearly independent.

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The following statements are equivalent

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- 2 The only solution to

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is 
$$t_1 = t_2 = \cdots = t_k = 0$$

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Any vector in the span of v<sub>1</sub>,..., v<sub>k</sub> can be expressed as a linear combination of v<sub>1</sub>,..., v<sub>k</sub> in a unique way

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- Any vector in the span of v<sub>1</sub>,..., v<sub>k</sub> can be expressed as a linear combination of v<sub>1</sub>,..., v<sub>k</sub> in a unique way
- The matrix  $A = \begin{pmatrix} \vec{v_1} & \vec{v_2} & \dots & \vec{v_k} \end{pmatrix}$  has rank k
- So The matrix equation  $A\vec{x} = \vec{0}$  has only the trivial solution,  $\vec{x} = \vec{0}$ .

$$(1) = 7 (2) \leftrightarrow (2)$$

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$$(3) \leftarrow (4)$$

# Proof

-

### More Proof

(y => (4) 5 is written wighty => A= (V, ... Ke) has rack K. A(b) (V. ... Vic) (t) > Vibit-...+ Victure. Claiming that the matrix A has a unique homagenery solution (in AFF= C => x=0) # for variables of t= # coloms - rk(4) => rk(4) = # colomny = k. and so has a unique home genous salution. (5)=)(1) A hog uniger home callet => Vi... Ku an lin ind. (net(1) => net(5)) if Vi... Ku we not lin inid the there exists a a Vi sto Vi= tivi + ... + to, Vin + tot, Vitit ... + tock O= tivit + - + tor Vir - Vit tot. Vitit + - + bave 

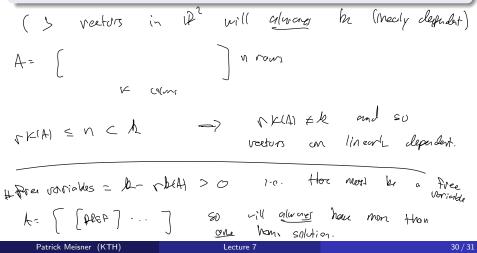
#### Theorem

The set of vectors  $\vec{v}_1, \ldots, \vec{v}_n$  in  $\mathbb{R}^n$  are linearly independent if and only if the matrix  $A = (\vec{v}_1 \quad \vec{v}_2 \quad \ldots \quad \vec{v}_n)$  is invertible.



#### Theorem

Suppose k > n. Then any set of k vectors  $\vec{v_1}, \ldots, \vec{v_k}$  in  $\mathbb{R}^n$  are linearly dependent.



### Geometric Interpretation

What does it mean geometrically for the set of two vectors  $\vec{v}, \vec{w}$  in  $\mathbb{R}^n$  to be linearly dependent?  $\vec{v} = \pm \vec{w} \implies 5$  some the  $\vec{v} = \pm \vec{w} \implies 5$  some line.  $\vec{v}$  colinear or period ortional

What does it mean *geometrically* for the set of three vectors  $\vec{u}, \vec{v}, \vec{w}$  in  $\mathbb{R}^n$  to be linearly dependent?