SF 1684 Algebra and Geometry
Lecture 7

Patrick Meisner

KTH Royal Institute of Technology

Topics for Today

1. Easily Invertible Matrices
2. Functions on Matrices: Transpose and Trace
3. Subspaces
4. Linear Dependence

Easily Invertible Matrices: Diagonal Matrices

Let

$$D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$$

be a diagonal matrix.

Easily Invertible Matrices: Diagonal Matrices

Let
$$D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$$
be a diagonal matrix. Then D is invertible if and only all of the d_i are non-zero

Easily Invertible Matrices: Diagonal Matrices

Let

$$D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$$

be a diagonal matrix. Then D is invertible if and only all of the d_i are non-zero with inverse

$$D^{-1} = \begin{pmatrix} \frac{1}{d_1} & 0 & \dots & 0 \\ 0 & \frac{1}{d_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{d_n} \end{pmatrix}$$

Check that

$DD^{-1} = I_n$

Easily Ivertible Matrices: 2×2 Matrix

In general, it is difficult to calculate the inverse of a given matrix.

Easily Ivertible Matrices: 2×2 Matrix

In general, it is difficult to calculate the inverse of a given matrix.

However, in the 2×2 case, there is a simple formula:

Theorem

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *then* A *is invertible if and only if* $ad - bc \neq 0$ *with inverse*

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Check: $\frac{1}{ad-bc} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} ad - bc & -ba + ab \\ dc - cd & -bc + ad \end{pmatrix}$

$= \frac{1}{ad-bc} \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Functions on Matrices

As well as multiplying and inverting matrices there are some other functions on matrices that we care about

Functions on Matrices

As well as multiplying and inverting matrices there are some other functions on matrices that we care about

Definition

Let A be an $m \times n$ matrix, then the **transpose** of A, denoted A^T is an $n \times m$ matrix, where the rows and columns are "flipped":

Functions on Matrices

As well as multiplying and inverting matrices there are some other functions on matrices that we care about

Definition

Let A be an $m \times n$ matrix, then the **transpose** of A, denoted A^T is an $n \times m$ matrix, where the rows and columns are "flipped":

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} \implies A^T = \begin{pmatrix} a_{1,1} & a_{2,1} & \cdots & a_{m,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \cdots & a_{m,n} \end{pmatrix}$$

Functions on Matrices

As well as multiplying and inverting matrices there are some other functions on matrices that we care about

Definition

Let A be an $m \times n$ matrix, then the **transpose** of A, denoted A^T is an $n \times m$ matrix, where the rows and columns are "flipped":

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix} \implies A^T = \begin{pmatrix} a_{1,1} & a_{2,1} & \dots & a_{m,1} \\ a_{1,2} & a_{2,2} & \dots & a_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \dots & a_{m,n} \end{pmatrix}$$

Example:

$$A := \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \implies A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Properties of Transposes

Theorem

Let A, B be matrices (of suitable dimensions) and c a real number

Properties of Transposes

Theorem

Let A, B be matrices (of suitable dimensions) and c a real number

1. $(A^T)^T = A$

Properties of Transposes

Theorem

Let A, B be matrices (of suitable dimensions) and c a real number

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$

Properties of Transposes

Theorem

Let A, B be matrices (of suitable dimensions) and c a real number

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(cA)^T = c(A^T)$

Properties of Transposes

Theorem

Let A, B be matrices (of suitable dimensions) and c a real number

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(cA)^T = c(A^T)$
4. $(AB)^T = B^T A^T$

Exercise: prove this

Hint: expand out each product carefully.

Properties of Transposes

Theorem

Let A, B be matrices (of suitable dimensions) and c a real number

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(cA)^T = c(A^T)$
4. $(AB)^T = B^T A^T$
5. $(A^T)^{-1} = (A^{-1})^T$

$\rightarrow$ Want to show that $\underline{A^T (A^{-1})^T = I}$

$\Downarrow$

$(A^{-1})^T = (A^T)^{-1}$

$$A^T (A^{-1})^T \overset{(4)}{=} (A^{-1} A)^T$$

$$= (I)^T = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}^T = I$$

Properties of Transposes

Theorem

Let A, B be matrices (of suitable dimensions) and c a real number

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(cA)^T = c(A^T)$
4. $(AB)^T = B^T A^T$
5. $(A^T)^{-1} = (A^{-1})^T$

IMPORTANT!!!!

Properties of Transposes

Theorem

Let A, B be matrices (of suitable dimensions) and c a real number

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(cA)^T = c(A^T)$
4. $(AB)^T = B^T A^T$
5. $(A^T)^{-1} = (A^{-1})^T$

IMPORTANT!!!!

$(AB)^T = B^T A^T$ and NOT $A^T B^T$

Properties of Transposes

Theorem

Let A, B be matrices (of suitable dimensions) and c a real number

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(cA)^T = c(A^T)$
4. $(AB)^T = B^T A^T$
5. $(A^T)^{-1} = (A^{-1})^T$

A is $m \times n$ so that
B is $n \times k$ AB makes sense

A^T is $n \times m$ so $B^T A^T$ makes sense
B^T is $k \times n$ $A^T B^T$ may not makes sense

IMPORTANT!!!!

$(AB)^T = B^T A^T$ and NOT $A^T B^T$ just like $(AB)^{-1} = B^{-1}A^{-1}$ and NOT $A^{-1}B^{-1}$.

Properties of Transposes

Theorem

Let A, B be matrices (of suitable dimensions) and c a real number

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(cA)^T = c(A^T)$
4. $(AB)^T = B^T A^T$
5. $(A^T)^{-1} = (A^{-1})^T$

IMPORTANT!!!!

$(AB)^T = B^T A^T$ and NOT $A^T B^T$ just like $(AB)^{-1} = B^{-1}A^{-1}$ and NOT $A^{-1}B^{-1}$.

The Dot Product as a Matrix Product

Let $\vec{v}$ be a vector in $\mathbb{R}^n$.

The Dot Product as a Matrix Product

Let $\vec{v}$ be a vector in $\mathbb{R}^n$. Then we can think about it as an $n \times 1$ matrix.

The Dot Product as a Matrix Product

Let $\vec{v}$ be a vector in $\mathbb{R}^n$. Then we can think about it as an $n \times 1$ matrix. Thus $\vec{v}^T$ is a $1 \times n$ matrix and their dimensions work out that we can multiply them.

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \qquad \vec{v}^T = \begin{pmatrix} v_1 \cdots & v_n \end{pmatrix}$$

The Dot Product as a Matrix Product

Let $\vec{v}$ be a vector in $\mathbb{R}^n$. Then we can think about it as an $n \times 1$ matrix. Thus $\vec{v}^T$ is a $1 \times n$ matrix and their dimensions work out that we can multiply them.

Aside

Theorem

Let $\vec{u}, \vec{v}$ be two vectors in $\mathbb{R}^n$. Then

$$\vec{v}^T \vec{u} = \vec{v} \cdot \vec{u}$$

$\vec{v} \cdot \vec{u} = \vec{v}\,\vec{u}$??

as matrices

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \qquad u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

$$v^T u = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

$$= v_1 u_1 + v_2 u_2 + \cdots + v_n u_n$$

$$= \vec{v} \cdot \vec{u}$$

v is $n \times 1$

u is $n \times 1$

$v * u$ (as matrices) does not make sense

The Trace Function

Definition

For a square matrix $A = (a_{i,j})$, we define the **trace** of the matrix as the sum of its diagonal entries:

$$\mathrm{Tr}(A) = a_{1,1} + a_{2,2} + \cdots + a_{n,n}$$

The Trace Function

Definition

For a square matrix $A = (a_{i,j})$, we define the **trace** of the matrix as the sum of its diagonal entries:

$$\mathrm{Tr}(A) = a_{1,1} + a_{2,2} + \cdots + a_{n,n}$$

Example:

$$\mathrm{Tr}\left(\begin{pmatrix} 3 & 6 & -1000 & 2 \\ 9001 & 1 & 44 & 54 \\ 0 & 789134 & 1 & 98 \\ -578 & 913 & 1 & 2 \end{pmatrix}\right) = 3 + 1 + 1 + 2 = 7$$

Properties of the Trace

Theorem

Let A and B be $n \times n$ square matrices and c any real number

Properties of the Trace

Theorem

Let A and B be $n \times n$ square matrices and c any real number

1. $Tr(cA) = c\,Tr(A)$

Properties of the Trace

Theorem

Let A and B be $n \times n$ square matrices and c any real number

1. $Tr(cA) = c\,Tr(A)$
2. $Tr(A + B) = Tr(A) + Tr(B)$

Properties of the Trace

Theorem

Let A and B be $n \times n$ square matrices and c any real number

1. $Tr(cA) = c\,Tr(A)$
2. $Tr(A + B) = Tr(A) + Tr(B)$
3. $Tr(A^T) = Tr(A)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Properties of the Trace

Theorem

Let A and B be $n \times n$ square matrices and c any real number

1. $Tr(cA) = c\,Tr(A)$
2. $Tr(A + B) = Tr(A) + Tr(B)$
3. $Tr(A^T) = Tr(A)$
4. $Tr(AB) = Tr(BA)$

Exercise: prove this

hint: expand everything with matrix multiplication.

Properties of the Trace

Theorem

Let A and B be $n \times n$ square matrices and c any real number

1. $Tr(cA) = c\,Tr(A)$
2. $Tr(A + B) = Tr(A) + Tr(B)$
3. $Tr(A^T) = Tr(A)$
4. $Tr(AB) = Tr(BA)$

Note: While it is almost never true that $AB = BA$, it happens that it is always true that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

Dot Product as a Trace

Recall that if $\vec{u}, \vec{v}$ are vectors in $\mathbb{R}^n$, then we can think of them as $n \times 1$ matrix and $\vec{v}^T$ as a $1 \times n$ matrix.

Dot Product as a Trace

Recall that if $\vec{u}, \vec{v}$ are vectors in $\mathbb{R}^n$, then we can think of them as $n \times 1$ matrix and $\vec{v}^T$ as a $1 \times n$ matrix. $\vec{v}^T \vec{u}$ then makes sense and is the dot product.

Dot Product as a Trace

Recall that if $\vec{u}, \vec{v}$ are vectors in $\mathbb{R}^n$, then we can think of them as $n \times 1$ matrix and $\vec{v}^T$ as a $1 \times n$ matrix. $\vec{v}^T \vec{u}$ then makes sense and is the dot product. But $\vec{v}\vec{u}^T$ also makes sense. What is this?

$n \times 1$ $\quad$ $1 \times n$

Dot Product as a Trace

Recall that if $\vec{u}, \vec{v}$ are vectors in $\mathbb{R}^n$, then we can think of them as $n \times 1$ matrix and $\vec{v}^T$ as a $1 \times n$ matrix. $\vec{v}^T \vec{u}$ then makes sense and is the dot product. But $\vec{v}\vec{u}^T$ also makes sense. What is this?

Theorem

Let $\vec{u}, \vec{v}$ be two matrices in $\mathbb{R}^n$. Then $\vec{v}\vec{u}^T$ is a square $n \times n$ matrix and

$$Tr(\vec{v}\vec{u}^T) = \vec{v} \cdot \vec{u}$$

$$V = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \qquad u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

$$V u^T = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \begin{pmatrix} u_1 & \dots & u_n \end{pmatrix} = \begin{bmatrix} u_1 v_1 & u_2 v_1 & \cdots & u_n v_1 \\ u_1 v_2 & u_2 v_2 & \cdots & u_n v_2 \\ \vdots & \vdots & & \vdots \\ u_1 v_n & u_2 v_n & \cdots & u_n v_n \end{bmatrix}$$

$$Tr(V u^T) = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

$$= \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$

Subspaces

Definition

A nonempty subset W of vectors in $\mathbb{R}^n$ is called a **subsapce** of $\mathbb{R}^n$ if it is closed under scalar multiplication and additions.

Subspaces

Definition

A nonempty subset W of vectors in $\mathbb{R}^n$ is called a **subsapce** of $\mathbb{R}^n$ if it is closed under scalar multiplication and additions. That is,

1. If $\vec{u} \in W$ and $c \in \mathbb{R}$, then $c\vec{u} \in W \Leftarrow$ closed under scalar multiplication.

$W \subseteq \mathbb{R}^n$

Subspaces

Definition

A nonempty subset W of vectors in $\mathbb{R}^n$ is called a **subsapce** of $\mathbb{R}^n$ if it is closed under scalar multiplication and additions. That is,

1. If $\vec{u} \in W$ and $c \in \mathbb{R}$, then $c\vec{u} \in W$ $\Leftarrow$ closed under scalar mult.
2. If $\vec{u}, \vec{v} \in W$ then $\vec{u} + \vec{w} \in W$. $\Leftarrow$ closed under addition

Subspaces

Definition

A nonempty subset W of vectors in $\mathbb{R}^n$ is called a **subsapce** of $\mathbb{R}^n$ if it is closed under scalar multiplication and additions. That is,

1. If $\vec{u} \in W$ and $c \in \mathbb{R}$, then $c\vec{u} \in W$
2. If $\vec{u}, \vec{v} \in W$ then $\vec{u} + \vec{w} \in W$.

Remark: if W is a subspace of $\mathbb{R}^n$ then it is *also* a vector space.

Exercise: go through the axioms and prove this.

Subspaces

Definition

A nonempty subset W of vectors in $\mathbb{R}^n$ is called a **subsapce** of $\mathbb{R}^n$ if it is closed under scalar multiplication and additions. That is,

1. If $\vec{u} \in W$ and $c \in \mathbb{R}$, then $c\vec{u} \in W$
2. If $\vec{u}, \vec{v} \in W$ then $\vec{u} + \vec{w} \in W$.

Remark: if W is a subspace of $\mathbb{R}^n$ then it is *also* a vector space.

Exercise

If W is a subspace of $\mathbb{R}^n$, then show that $\vec{0} \in W$.

W is nonempty so there exists a $\vec{u} \in W$

W is closed under scalar multiplication so $(-1) \cdot \vec{u} \in W$

W is closed under addition so $\vec{u} + (-1) \cdot \vec{u} \in W$

But we've seen previously that $(-1)\vec{u} = -\vec{u}$ so $\vec{0} = u + -\vec{u} \in W$

Smallest Subspace

Definition

Let $W = \{\vec{0}\}$ be the set containing only the zero vector of $\mathbb{R}^n$. W is called the **zero subspace**.

Smallest Subspace

Definition

Let $W = \{\vec{0}\}$ be the set containing only the zero vector of $\mathbb{R}^n$. W is called the **zero subspace**.

Exercise

Show that the zero subspace actually *is* a subspace of $\mathbb{R}^n$.

Check: W is nonempty: $\vec{0} \in W$

check: closed under scalar mult: $c \in \mathbb{R}$ $\vec{v} \in W \Rightarrow \vec{v} = \vec{0}$

and so $c \cdot \vec{v} = c \cdot \vec{0} = \vec{0} \in W$

check: closed under addition: $\vec{u}, \vec{v} \in W \Rightarrow \vec{u} = \vec{v} = \vec{0}$

$\vec{u} + \vec{v} = \vec{0} + \vec{0} = \vec{0} \in W$

Homogenous Subspace

Theorem

The set of homogeneous solutions to a matrix A is a subspace of $\mathbb{R}^n$.

Homogenous Subspace

Theorem

The set of homogeneous solutions to a matrix A is a subspace of $\mathbb{R}^n$. ~~If~~ A is invertible, ~~then~~ iff this subspace is the zero subspace.

check: non-empty. $\vec{0}$ is always a homogeneous solution.

check: scalar mult. $\vec{x}$ homo. sol $c \in \mathbb{R}$ $A(c\vec{x}) = c(A\vec{x}) = c \cdot \vec{0} = \vec{0}$

so $c\vec{x}$ is homo. solution.

check: addition: $\vec{x}, \vec{y}$ are homo sol. $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0}$

so $\vec{x}+\vec{y}$ is homo solution

$\Longrightarrow$ the set of homo solutions is a subspace.

Showed last class that A is invertible iff $A\vec{x} = 0$ has a unique solution namely $\vec{x} = \vec{0}$.

Characterisation of Zero Matrix

Theorem

If A is a matrix with n columns then the subspace of homogeneous solutions is all of $\mathbb{R}^n$ if and only if $A = 0$, the zero matrix.

Characterisation of Zero Matrix

Theorem

If A is a matrix with n columns then the subspace of homogeneous solutions is all of $\mathbb{R}^n$ if and only if $A = 0$, the zero matrix. i.e. $A\vec{x} = 0$ for all $\vec{x} \in \mathbb{R}^n$ if and only if $A = 0$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & & \vdots \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix}$$

0

$$\vec{0} = A\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} \Rightarrow a_{11} = 0, \ldots, a_{m1} = 0$$

$$\vec{0} = A\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{bmatrix} \Rightarrow a_{1i} = 0 \quad \cdots \quad a_{mi} = 0$$

1 is a i^{th} position

hence all $a_{ij} = 0$ and so $A = 0$.

Characterisation of Matrices

Theorem

If A and B are matrices with n columns, then $A\vec{x} = B\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$ if and only if $A = B$.

If $A\vec{x} = B\vec{y}$ for all $x \in \mathbb{R}^n$ then

$(A-B)\vec{x} = Ax - Bx = \vec{0}$ for all $x \in \mathbb{R}^n$

$\Rightarrow A - B = 0 \Rightarrow A = B.$

Next Simplest Subspace

Theorem

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be vectors in $\mathbb{R}^n$, then the set of all linear combinations of them

$$W = \{\vec{x} = t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k : t_1, t_2, \ldots, t_k \in \mathbb{R}\}$$

is a subspace of $\mathbb{R}^n$

check: W is non-empty: obvious as choice of $t_1 \ldots t_k \in \mathbb{R}$

check: scalar multiplication $\vec{x} \in W$ $c \in \mathbb{R}$

$$c \cdot \vec{x} = c\left(t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k\right) = \underbrace{(ct_1)\vec{v}_1 + \cdots + (ct_k)\vec{v}_k}_{\in W}$$

check: addition: $\vec{x}, \vec{y} \in W$

$$\vec{x} + \vec{y} = \left(t_1\vec{v}_1 + \cdots + t_k\vec{v}_k\right) + \left(s_1\vec{v}_1 + \cdots + s_k\vec{v}_k\right) = (t_1+s_1)\vec{v}_1 + \cdots + (t_k+s_k)\vec{v}_k \in W.$$

Span of Set of Vectors

The example in the previous slide is called the **span** of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$. We often denote it

$$\begin{aligned} W &= \{\vec{x} = t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k : t_1, t_2, \ldots, t_k \in \mathbb{R}\} \\ &= \operatorname{span}_{\mathbb{R}}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \quad \rightarrow \text{ span ove the field of the real numbers.} \\ &= \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \end{aligned}$$

and we call the set of vectors $\vec{v}_1, \ldots, \vec{v}_k$ a **spanning set of vectors** for W.

Span of Set of Vectors

The example in the previous slide is called the **span** of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$. We often denote it

$$\begin{aligned} W &= \{\vec{x} = t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k : t_1, t_2, \dots, t_k \in \mathbb{R}\} \\ &= \operatorname{span}_{\mathbb{R}} \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\} \\ &= \operatorname{span} \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\} \end{aligned}$$

and we call the set of vectors $\vec{v}_1, \dots, \vec{v}_k$ a **spanning set of vectors** for W.

Spanning Set of Vectors for $\mathbb{R}^n$

Recall we have the **standard normal vectors** of $\mathbb{R}^n$

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \dots \quad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Spanning Set of Vectors for $\mathbb{R}^n$

Recall we have the **standard normal vectors** of $\mathbb{R}^n$

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \dots \quad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

then we can write

$$\mathbb{R}^n = \operatorname{span}\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$$

Spanning Set of Vectors for $\mathbb{R}^n$

Recall we have the **standard normal vectors** of $\mathbb{R}^n$

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \dots \quad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

then we can write

$$\mathbb{R}^n = \text{span}\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$$

So $\vec{e}_1, \dots, \vec{e}_n$ is a spanning set of vectors for $\mathbb{R}^n$

Exercise

Exercise

Find the subspace of homogeneous solutions of

$$A = \begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{pmatrix}$$

and write it as a span of vectors.

Exercise

Exercise

Find the subspace of homogeneous solutions of

$$A = \begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{pmatrix}$$

and write it as a span of vectors.

We know that set of homogeneous solutions will all those $\vec{x}$ that satisfy $A\vec{x} = \vec{0}$.

Exercise

Find the subspace of homogeneous solutions of

$$A = \begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{pmatrix}$$

and write it as a span of vectors.

We know that set of homogeneous solutions will all those $\vec{x}$ that satisfy $A\vec{x} = \vec{0}$. To solve this we reduce A down to RREF

$$\begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{pmatrix}$$

Exercise

Exercise

Find the subspace of homogeneous solutions of

$$A = \begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{pmatrix}$$

and write it as a span of vectors.

We know that set of homogeneous solutions will all those $\vec{x}$ that satisfy $A\vec{x} = \vec{0}$. To solve this we reduce A down to RREF

$$\begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 0 \\ 0 & 0 & 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence, we need to find the set of $\vec{x}$ such that

$$\begin{array}{c} \begin{matrix} & x_2 & & x_4 & x_5 & \end{matrix} \\ \begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 0 \\ 0 & 0 & 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{matrix} & t_1 & & t_2 & t_3 & \end{matrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence, we need to find the set of $\vec{x}$ such that

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 0 \\ 0 & 0 & 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \implies$$

$$\implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \overset{v_1}{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} \underset{\text{par}}{t_1} + \overset{v_2}{\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}} \underset{\text{par}}{t_2} + \overset{v_3}{\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}} \underset{\text{par}}{t_3}$$

Exercise Continued

Therefore, we can conclude that the subspace of homogeneous solutions to A is

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} t_1 + \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} t_2 + \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} t_3 : t_1, t_2, t_3 \in \mathbb{R} \right\}$$

$$= \operatorname{span} \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Different Spanning Set of Vectors

We saw that

$$\mathbb{R}^n = \text{span}\{\vec{e}_1, \dots, \vec{e}_n\}$$

so that $\vec{e}_1, \dots, \vec{e}_n$ is a spanning set of vectors.

Different Spanning Set of Vectors

We saw that

$$\mathbb{R}^n = \text{span}\{\vec{e}_1, \ldots, \vec{e}_n\}$$

so that $\vec{e}_1, \ldots, \vec{e}_n$ is a spanning set of vectors. However, we can find many different spanning sets of vectors for the same vector space

$$\mathbb{R}^2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

Different Spanning Set of Vectors

We saw that

$$\mathbb{R}^n = \text{span}\{\vec{e}_1, \ldots, \vec{e}_n\}$$

so that $\vec{e}_1, \ldots, \vec{e}_n$ is a spanning set of vectors. However, we can find many different spanning sets of vectors for the same vector space

exercise: check this

$$\mathbb{R}^2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$$

Different Spanning Set of Vectors

We saw that

$$\mathbb{R}^n = \text{span}\{\vec{e}_1, \ldots, \vec{e}_n\}$$

so that $\vec{e}_1, \ldots, \vec{e}_n$ is a spanning set of vectors. However, we can find many different spanning sets of vectors for the same vector space

$$\mathbb{R}^2 = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\}$$

Different Spanning Set of Vectors

We saw that

$$\mathbb{R}^n = \text{span}\{\vec{e}_1, \dots, \vec{e}_n\}$$

so that $\vec{e}_1, \dots, \vec{e}_n$ is a spanning set of vectors. However, we can find many different spanning sets of vectors for the same vector space

$$\mathbb{R}^2 = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\}$$

However, we see there is some redundancy in that last example

Redundancy in Spanning Sets

In the last example we had

$$\mathbb{R}^2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

Redundancy in Spanning Sets

In the last example we had

$$\mathbb{R}^2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

However, we see that

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Redundancy in Spanning Sets

In the last example we had

$$\mathbb{R}^2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

However, we see that

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so any linear combination of all three of the vectors can easily be written as a linear combination of first two:

Redundancy in Spanning Sets

In the last example we had

$$\mathbb{R}^2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

However, we see that

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so any linear combination of all three of the vectors can easily be written as a linear combination of first two:

$$t_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t_3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Redundancy in Spanning Sets

In the last example we had

$$\mathbb{R}^2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

However, we see that

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so any linear combination of all three of the vectors can easily be written as a linear combination of first two:

$$t_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t_3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t_3 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

Redundancy in Spanning Sets

In the last example we had

$$\mathbb{R}^2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

However, we see that

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so any linear combination of all three of the vectors can easily be written as a linear combination of first two:

$$t_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t_3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t_3 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

$$= (t_1 + t_3) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (t_2 + t_3) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We then say that these three vectors are **linearly dependent**.

Linearly Dependent and Independent

Definition

We say that a set of vectors $\vec{v}_1, \ldots, \vec{v}_k$ are **linearly dependent** if there is a $\vec{v}_i$ that can be written as a linear combination of the rest of the vectors:

$$\vec{v}_i = t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_{i-1}\vec{v}_{i-1} + t_{i+1}\vec{v}_{i+1} + \cdots + t_k\vec{v}_k$$

v_i does not appear on the RHS.

Linearly Dependent and Independent

Definition

We say that a set of vectors $\vec{v}_1, \ldots, \vec{v}_k$ are **linearly dependent** if there is a $\vec{v}_i$ that can be written as a linear combination of the rest of the vectors:

$$\vec{v}_i = t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_{i-1}\vec{v}_{i-1} + t_{i+1}\vec{v}_{i+1} + \cdots + t_k\vec{v}_k$$

If no such relationship exists we say the vectors are **linearly independent**.

Linearly Dependent and Independent

Definition

We say that a set of vectors $\vec{v}_1, \ldots, \vec{v}_k$ are **linearly dependent** if there is a $\vec{v}_i$ that can be written as a linear combination of the rest of the vectors:

$$\vec{v}_i = t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_{i-1}\vec{v}_{i-1} + t_{i+1}\vec{v}_{i+1} + \cdots + t_k\vec{v}_k$$

If no such relationship exists we say the vectors are **linearly independent**.

Fact

Any set of vectors containing $\vec{0}$ is linearly dependent.

$\vec{v}_1 \ldots \vec{v}_k$ and $\vec{v}_i = \vec{0}$ then

$\vec{v}_i = \vec{0} = 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_{i-1} + 0 \cdot v_{i+1} + \cdots + 0 \cdot v_k$

Creating Linearly Independence out of Linearly Dependence

If $\vec{v}_1, \ldots, \vec{v}_k$ is a linearly dependent set of vectors with $\vec{v}_i$ being able to be written as a linear combination of the rest of the vectors, then we get

$$\text{span}\{\vec{v}_1, \ldots, \vec{v}_k\} = \text{span}\{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_k\}$$

$t_1 v_1 + \cdots + t_i v_i + \cdots t_n v_n = t_1 v_1 + \cdots + t_i(s_1 v_1 + \cdots + s_n v_n) + \cdots + \cancel{t_n} v_n$

v_i doesn't appear

$= (t_1 + t_i s_1) v_1 + \cdots + (t_n + t_i s_n) v_n$

v_i doesn't appear

Creating Linearly Independence out of Linearly Dependence

If $\vec{v}_1, \dots, \vec{v}_k$ is a linearly dependent set of vectors with $\vec{v}_i$ being able to be written as a linear combination of the rest of the vectors, then we get

$$\text{span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k\}$$

Hence, as long as the remaining vectors are still linearly dependent, then we can keep removing one vector without affecting the span of the vectors.

Creating Linearly Independence out of Linearly Dependence

If $\vec{v}_1, \ldots, \vec{v}_k$ is a linearly dependent set of vectors with $\vec{v}_i$ being able to be written as a linear combination of the rest of the vectors, then we get

$$\text{span}\{\vec{v}_1, \ldots, \vec{v}_k\} = \text{span}\{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_k\}$$

Hence, as long as the remaining vectors are still linearly dependent, then we can keep removing one vector without affecting the span of the vectors.

Theorem

For any set of vectors $\vec{v}_1, \ldots, \vec{v}_k$, we can find a subset of the vectors $\vec{v}_{i_1}, \ldots, \vec{v}_{i_\ell}$ such that

$$span\{\vec{v}_1, \ldots, \vec{v}_k\} = span\{\vec{v}_{i_1}, \ldots, \vec{v}_{i_\ell}\}$$

and $\vec{v}_{i_1}, \ldots, \vec{v}_{i_\ell}$ is linearly independent.

Linearly Independent Theorem

Theorem

The following statements are equivalent

Linearly Independent Theorem

Theorem

The following statements are equivalent

1. *The set of vectors* $\vec{v}_1, \ldots, \vec{v}_k$ *in* $\mathbb{R}^n$ *are* **linearly independent**

Linearly Independent Theorem

Theorem

The following statements are equivalent

1. *The set of vectors* $\vec{v}_1, \ldots, \vec{v}_k$ *in* $\mathbb{R}^n$ *are* **linearly independent**
2. *The only solution to*

$$t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k = \vec{0}$$

is $t_1 = t_2 = \cdots = t_k = 0$

Linearly Independent Theorem

Theorem

The following statements are equivalent

1. *The set of vectors* $\vec{v}_1, \ldots, \vec{v}_k$ *in* $\mathbb{R}^n$ *are* **linearly independent**
2. *The only solution to*

$$t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k = \vec{0}$$

   is $t_1 = t_2 = \cdots = t_k = 0$
3. *Any vector in the span of* $\vec{v}_1, \ldots, \vec{v}_k$ *can be expressed as a linear combination of* $\vec{v}_1, \ldots, \vec{v}_k$ *in a* **unique** *way*

Linearly Independent Theorem

Theorem

The following statements are equivalent

1. *The set of vectors* $\vec{v}_1, \dots, \vec{v}_k$ *in* $\mathbb{R}^n$ *are* **linearly independent**
2. *The only solution to*

$$t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k = \vec{0}$$

   is $t_1 = t_2 = \cdots = t_k = 0$
3. *Any vector in the span of* $\vec{v}_1, \dots, \vec{v}_k$ *can be expressed as a linear combination of* $\vec{v}_1, \dots, \vec{v}_k$ *in a* **unique** *way*
4. *The matrix* $A = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_k \end{pmatrix}$ *has rank* k

Linearly Independent Theorem

Theorem

The following statements are equivalent

1. *The set of vectors* $\vec{v}_1, \ldots, \vec{v}_k$ *in* $\mathbb{R}^n$ *are* **linearly independent**
2. *The only solution to*
$$t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k = \vec{0}$$
is $t_1 = t_2 = \cdots = t_k = 0$
3. *Any vector in the span of* $\vec{v}_1, \ldots, \vec{v}_k$ *can be expressed as a linear combination of* $\vec{v}_1, \ldots, \vec{v}_k$ *in a* **unique** *way*
4. *The matrix* $A = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_k \end{pmatrix}$ *has rank* k
5. *The matrix equation* $A\vec{x} = \vec{0}$ *has only the trivial solution,* $\vec{x} = \vec{0}$.

$$\begin{tikzcd} (1) \arrow[r, Rightarrow] & (2) \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & (3) \\ (5) \arrow[u, Rightarrow] & (4) \arrow[l, Rightarrow] & \end{tikzcd}$$

Proof

(1) ⇒ (2) (not (2) ⇒ not (1))

Suppose there is a solution $t_1 v_1 + \dots + t_k v_k = 0$ with not all $t_i = 0$

The choose one of the t_i's that are not 0 and bring it to RHS:

$$t_1 v_1 + \dots + t_{i-1} v_{i-1} + t_{i+1} v_{i+1} + \dots + t_k v_k = -t_i v_i$$

$$v_i = \frac{-t_1}{t_i} v_1 + \dots + \frac{-t_{i-1}}{t_i} v_{i-1} + \frac{-t_{i+1}}{t_i} v_{i+1} + \dots + \frac{-t_k}{t_i} v_k$$

not linearly independent.

(2) ⇒ (3) (not (3) ⇒ not (2)) $w \in$ span $(v_1 \dots v_k)$ and can be expressed in two different ways $w = t_1 v_1 + \dots + t_k v_k = s_1 v_1 + \dots + s_k v_k$ when not all t_i are equal to s_i.

$\vec{0} = w - w = t_1 v_1 + \dots + t_k v_k - s_1 v_1 \dots - s_k v_k = (t_1 - s_1) v_1 + \dots + (t_k - s_k) v_k$

$t_1 - s_1 \dots t_k - s_k$ are not all 0.

(3) ⇒ (2) obvious. If every vector can be written uniquely then $\vec{0}$ can be written uniquely

More Proof

(4) ⇒ (4) $\vec{0}$ is written uniquely ⇒ $A = (V_1 \cdots V_k)$ has rank k.

$A\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} = (V_1 \cdots V_k)\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} = V_1 t_1 + \cdots + V_k t_k$. Claiming that the matrix

A has a unique homogeneous solution (i.e. $A\vec{x} = 0 \Rightarrow \vec{x} = 0$)

$\#$ free variables of $A = \underset{(k)}{\#\text{ columns}} - rk(A)$. $\Rightarrow rk(A) = \#$ columns $= k$.

$\overset{0}{\rule{0pt}{0pt}}$

(4) ⇒ (5) if A has rank k then $\#$ free variables $= 0$

and so has a unique homogeneous solution.

(5) ⇒ (1) A has unique homo solution ⇒ $V_1 \cdots V_k$ are lin ind.

(not(1) ⇒ not(5)) if $V_1 \cdots V_k$ are not lin. ind then there exists a

a V_i s.t. $V_i = t_1 V_1 + \cdots + t_{i-1} V_{i-1} + t_{i+1} V_{i+1} + \cdots + t_k V_k$

$0 = t_1 V_1 + \cdots + t_{i-1} V_{i-1} - V_i + t_{i+1} V_{i+1} + \cdots + t_k V_k$

$A\begin{pmatrix} t_1 \\ \vdots \\ -1 \\ \vdots \\ t_k \end{pmatrix} = \vec{0}$ so $\begin{pmatrix} t_1 \\ \vdots \\ -1 \\ \vdots \\ t_k \end{pmatrix}$ is a homogeneous sol but not the trivial solution.

$k = n$

Theorem

The set of vectors $\vec{v}_1, \dots, \vec{v}_n$ in $\mathbb{R}^n$ are linearly independent if and only if the matrix $A = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{pmatrix}$ is invertible.

$v_1 \cdots v_n$ are lin indep $\Longleftrightarrow$ A has rank n

$\overset{\text{previous thm}}{\Longleftrightarrow}$ A has RREF I_n

$\Longleftrightarrow$ A is invertible.

$k > n$

Theorem

Suppose $k > n$. Then any set of k vectors $\vec{v}_1, \ldots, \vec{v}_k$ in $\mathbb{R}^n$ are linearly dependent.

(3 vectors in $\mathbb{R}^2$ will always be linearly dependent)

$$A = \underbrace{\left[\qquad\qquad\qquad \right]}_{k \text{ colums}} \; n \text{ rows}$$

$rk(A) \le n < k \implies rk(A) \ne k$ and so vectors are linearly dependent.

free variables $= k - rk(A) > 0$ i.e. there must be a free variable

$A = \left[\left[RREF \right] \ldots \right]$ so will always have more than one hom. solution.

What does it mean *geometrically* for the set of two vectors $\vec{v}, \vec{w}$ in $\mathbb{R}^n$ to be linearly dependent?

$\vec{v} = t\vec{w} \Rightarrow$ $\vec{v}$ & $\vec{w}$ are on the same line.

or colinear or parallel or proportional

What does it mean *geometrically* for the set of three vectors $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^n$ to be linearly dependent?

$\vec{u} = t_1\vec{v} + t_2\vec{w} \Rightarrow$ $\vec{u}$ lies on the same plane as $\vec{v}$ & $\vec{w}$

$\vec{u}, \vec{v}, \vec{w}$ are coplanar