# SF 1684 Algebra and Geometry Lecture 6

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- **1** Using Matrix Multiplication to Solve  $A\vec{x} = \vec{b}$
- 2 Matrix Inverse
- Ilementary Matrices

# Solving Matrix Equations

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$$ax = b$$
  
divide both sides by q:  $X = \frac{b}{q}$ 

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Motivating example: if a and b were real numbers (or vectors in  $\mathbb{R}^1$ ), how would we solve

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Today, will be devoted to developing a way to "divide matrices". But first, we should understand fully what it means to multiply matrices.

Last time we showed that the set of  $m \times n$  matrices is a vector space and so behaves well with scalar multiplication. i.e.

$$\underline{c}(A+B) = \underline{c}A + \underline{c}B \qquad (\underline{c}+\underline{d})A = \underline{c}A + \underline{d}A \cdots \qquad \forall c \in \mathbb{Z}$$

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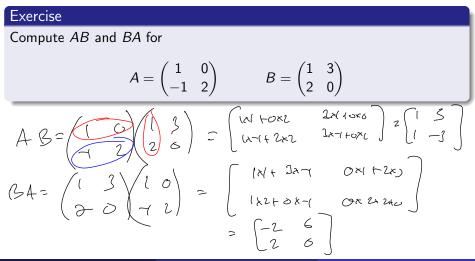
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•  $A(B \pm C) = AB \pm AC$   
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•  $c(BC) = (cB)C = B(cC)$ 

# Non Commutative

### CAUTION!!!!!

Even if both make sense, in general  $AB \neq BA!!!$ 

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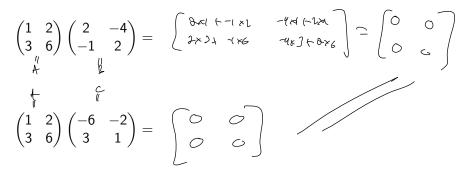
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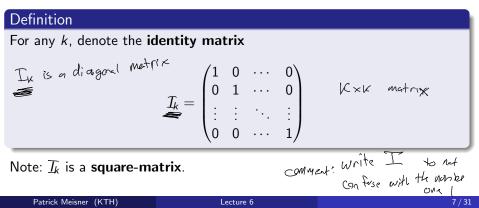
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Exercise

Compute  $AI_3$  and  $I_2A$  for

$$A = \begin{pmatrix} -8 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix}$$

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# Solving $(I_n | \vec{b})$ Using Matrix Multiplication

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However, we can view  $\vec{x}$  as an  $n \times 1$  matrix and so get that

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from which we can conclude that  $\vec{x}$  solves  $(I_n | \vec{b})$  if and only if

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## A is nxu

#### Theorem

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- If  $R = I_n$ , then the system has a unique solution for  $\mathcal{A}(\mathcal{A}|\mathcal{b})$
- If R ≠ I<sub>n</sub> then rk(A) < n and the system has either 0 or infinitely many solutions.</p>

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Recall: we say that two matrices are **row equivalent** if one can be obtained from the other from a series of row operations. So the condition  $R = I_n$  can be rephrased as A is row equivalent to  $I_n$  and  $R \neq I_n$  can be rephrased as A is *not* row equivalent to  $I_n$ .

# Proof

 $\mathbb{O}(A|\tilde{b}) \xrightarrow{\text{formedia}} (R|\tilde{c}) = (I_n|\tilde{c})$  $\chi$  solves  $(A(\vec{b}))$  iff  $\vec{x}$  solve  $(\pm (\vec{c}))$  iff  $\vec{x} = \vec{c}$ (2) (A(5) (P(c)) (R(c)) (For the variable = N-rK(R)) (For the variable > 0) (For the variable > 0) (For the leading Li con be So I much have a leading ones but n rows so by extradistion & must have a row of zerzs, R= -. Of C = C => then mult be a free variable => millions

Patrick Meisner (KTH)

ax = b

$$ax = b \implies \frac{ax}{a} = \frac{b}{a}$$

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Question

# But what is "division by a"?

$$ax = b \implies \frac{ax}{a} = \frac{b}{a} \implies \frac{a}{a}x = \frac{b}{a} \implies 1x = \frac{b}{a}$$

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Division is defined as the opposite of multiplication.

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```
"dividing by a" is the same as "multiplying by \frac{1}{2}"
```

# Simplest Example 2

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## Conclusion

To solve ax = b it is best to think about *multiplying by the multiplicative inverse* of *a* then to think about dividing by *a*.

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$$\implies \vec{x} = \begin{bmatrix} -7\\ 5 \end{bmatrix}$$

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This definition if for square matrices only (m = n). We can not talk about inverse of non-square matrices!

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In particular, there is exactly one solution to the augmented matrix  $(A|\vec{b})$  for all  $\vec{b}$ .

$$(-7)Ax=b$$
 Multiply both sides by  $A^{7}$ :  $A^{7}Ax = A^{7}b$   
->  $I_{n}x = A^{7}b$   
 $x = A^{7}b$ 

$$(=)$$
  $\tilde{X} = A^{-1}b$  He  $f\tilde{X} = AA^{-1}b = I_{11}b = \bar{b}$ 

#### Theorem

Let A be an  $n \times n$  matrix. The the following are equivalent

•  $A\vec{x} = \vec{b}$  has a unique solution for every  $\vec{b}$ 

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- A is invertible

$$(1) \Rightarrow (21 = (2) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$$

$$(1) \Rightarrow (2) (f Ax = 6 has a colution for all 6 then in putiendo for  $5x$   
(2) = (3) Ax > 0 dummer has a solution of  $x = 0$   
so Ax > 0 hos 4 solution of hos solutions.  
(f it has a force variable then  $hK(f) = n - # free variable < N$$$

### More Work Space

$$\frac{(2)^{-7}(2)}{Hn} \xrightarrow{(2)^{-7}(2)} (2n+1) \xrightarrow{(2)^{-7}(2)^{-7}(2)} (2n+1) \xrightarrow{(2)^{-7}(2)^{-7}(2)} (2n+1) \xrightarrow{(2)^{-7}(2)^{-7}(2)} (2n+1) \xrightarrow{(2)^{-7}(2)^{-7}(2)^{-7}(2)} (2n+1) \xrightarrow{(2)^{-7}(2)^{$$

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- *cA* is invertible with  $(cA)^{-1} = \frac{1}{c}A^{-1}$

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• cA is invertible with 
$$(cA)^{-1} = rac{1}{c}A^{-1}$$

**D**AB is invertible with  $(AB)^{-1} = B^{-1}A^{-1}$ .  $\frown$  biz one [

### More Work Space

3) Want to show (I'l'= A B(AT) is the Motrix such that A'B= (3A'= I we at that A=B satisfies this I by the uniquences is the only metrix that satisfies this (a)  $(CA)^{T} = \tilde{c}A^{-1}$  deck:  $(CA) \cdot (\tilde{c}A^{-1}) = C \cdot \tilde{c} \cdot A \cdot A^{T}$ =1.2=  $(AB)^{T} \neq A^{T}B^{T}$ BT AT (A-B) Ĵ cheak + hat (RT A') (AB) 2 B(A'A)B  $= B^{T}(IP) = B^{T}B = I$ 

Recall that the elementary row operation are

- Add a multiple of one row to the other
- Interchange two rows
- Multiply a row by a non-zero constant

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#### Definition

We say a matrix is an **elementary matrix** if it can be obtained by one elementary row operation performed on the identity matrix.

Add a multiple of one row to the other

Add a multiple of one row to the other

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 + 4R_1} \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \overleftarrow{(-)}$$

Add a multiple of one row to the other

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Interchange two rows

Add a multiple of one row to the other

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 + 4R_1} \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \simeq$$

Interchange two rows

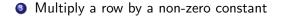
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \iff R_3} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \simeq \overbrace{\vdash}^{\sim}$$

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Multiply a row by a non-zero constant

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{-2R_1} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cong \bigcup$$

# Inverse of Elementary Matrices

#### Theorem

If E is an elementary matrix then it is invertible and  $E^{-1}$  is also an elementary matrix.

If E is an elementary matrix then it is invertible and  $E^{-1}$  is also an elementary matrix. Moreover,  $E^{-1}$  corresponds to the "undoing" row operation of E.

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$$F_{2} + 4 R_{1} \rightarrow \left( \begin{array}{c} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right)^{-1} = \left( \begin{array}{c} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \leftarrow F_{2} - 4 R_{1}$$

$$F_{3} \Rightarrow \left( \begin{array}{c} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array} \right)^{-1} = \left( \begin{array}{c} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array} \right) \leftarrow F_{3} - 4 R_{1}$$

$$F_{4} \Rightarrow \left( \begin{array}{c} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right)^{-1} = \left( \begin{array}{c} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \leftarrow -\frac{1}{2} R_{1}$$

Given a matrix A and an elementary matrix E, then the matrix EA is obtained by performing the row operation corresponding to E on A.

Given a matrix A and an elementary matrix E, then the matrix EA is obtained by performing the row operation corresponding to E on A.

$$\begin{array}{c} \begin{array}{c} & & & \\ & & \\ & \\ & \\ & \\ \end{array} \end{array} \xrightarrow{\ell} \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 6 & -1 & 11 & 18 \\ 1 & 4 & 4 & 0 \end{pmatrix}$$

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Note: It is only EA that corresponds to performing the row operation. AE does NOT correspond to this. In fact, AE may not even make sense!

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Note: It is only EA that corresponds to performing the row operation. AE does NOT correspond to this. In fact, AE may not even make sense!

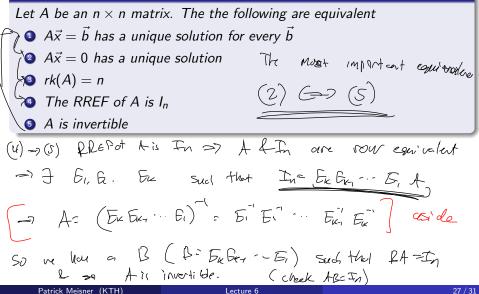
#### Theorem

Two matrices A and B are row equivalent if and only if there is a series of elementary matrices such that

$$B=E_kE_{k-1}\cdots E_2E_1A$$

# $(4) \implies (5)$

#### Theorem



One consequence of the above theorem is that A is invertible if and only if there is a series of row operations that reduce it to I.

$$E_k E_{k_1} \cdots E_2 E_1 A = I$$

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However, this now implies that

$$A^{-1} = E_k E_{k-1} \cdots E_2 E_1$$

$$E_k E_{k_1} \cdots E_2 E_1 A = I$$

However, this now implies that

$$A^{-1} = \underbrace{E_k E_{k-1} \cdots E_2 E_1}_{I_1}$$

$$E_k E_{k_1} \cdots E_2 E_1 A = I$$

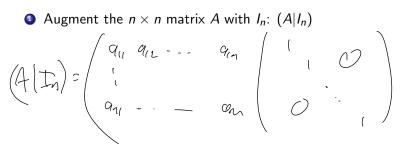
However, this now implies that

$$A^{-1} = E_k E_{k-1} \cdots E_2 E_1 I$$

#### Conclusion

To find the inverse of A, it is enough to apply row operation that reduce A to I on I itself.





- Augment the  $n \times n$  matrix A with  $I_n$ :  $(A|I_n)$
- 2 Perform Gauss-Jordan elimination on A while at the same time doing the same row operations to  $I_n$

$$(A|I_n) \implies (R|B)$$

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$$(A|I_n) \Longrightarrow (R|\underline{B})$$

• If the 
$$R = I_n$$
 then  $A$  is invertible and  $B = A^{-1}$ .

- Augment the  $n \times n$  matrix A with  $I_n$ :  $(A|I_n)$
- **②** Perform Gauss-Jordan elimination on A while at the same time doing the same row operations to  $I_n$

$$(A|I_n) \implies (R|B)$$

- If the  $R = I_n$  then A is invertible and  $B = A^{-1}$ .
- If  $R \neq I_n$  then A is not invertible.

# Finding the Inverse

Find the inverse of the matrix

$$A = \begin{pmatrix} -1 & 3 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

$$(A \mid I_{n}) = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 2 & 1 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 0 \\ 0 & 0$$

### More Work Space

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\end{array}\right)^{-(0)} + \left(\begin{array}{c}
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