SF 1684 Algebra and Geometry
Lecture 6

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Topics for Today

1. Using Matrix Multiplication to Solve $A\vec{x} = \vec{b}$
2. Matrix Inverse
3. Elementary Matrices

Solving Matrix Equations

The main topic for today is to develop a way to solve the matrix equation

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Today, will be devoted to developing a way to "divide matrices". But first, we should understand fully what it means to multiply matrices.

Properties of Matrix Multiplication

Last time we showed that the set of $m \times n$ matrices is a vector space and so behaves well with scalar multiplication. i.e.

$$\underline{c}(A+B) = \underline{c}A + \underline{c}B \qquad (\underline{c} + \underline{d})A = \underline{c}A + \underline{d}A \cdots \qquad c,d \in \mathbb{R}$$

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2. $A(B \pm C) = AB \pm AC$
3. $(B \pm C)A = BA \pm CA$
4. $c(BC) = (cB)C = B(cC)$

Non Commutative

CAUTION!!!!!

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Exercise

Compute AB and BA for

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix} = \begin{bmatrix} 1\times 1+0\times 2 & 2\times 1+0\times 0 \\ 1\times -1+2\times 2 & 3\times -1+0\times 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -3 \end{bmatrix}$$

$$BA = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} = \begin{bmatrix} 1\times 1+3\times -1 & 0\times 1+2\times 3 \\ 1\times 2+0\times -1 & 0\times 2+2\times 0 \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ 2 & 0 \end{bmatrix}$$

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1. $AB = AC$ does *NOT* imply $B = C$
2. $AB = 0$ does *NOT* imply $A = 0$ or $B = 0$

$$\underbrace{\begin{pmatrix}1 & 2\\ 3 & 6\end{pmatrix}}_{=A}\underbrace{\begin{pmatrix}2 & -4\\ -1 & 2\end{pmatrix}}_{=B} = \begin{bmatrix}2\times1+-1\times2 & -4\times1+2\times2\\ 2\times3+-1\times6 & -4\times3+2\times6\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\overbrace{\begin{pmatrix}1 & 2\\ 3 & 6\end{pmatrix}}^{A=}\overbrace{\begin{pmatrix}-6 & -2\\ 3 & 1\end{pmatrix}}^{C=} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Identity Matrix

We saw that the matrix of all zeros (the **zero matrix**) behaves like the number 0. i.e.

$$A + 0 = A \qquad A + (-A) = 0 \cdots$$

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Definition

For any k, denote the **identity matrix**

I_k is a diagonal matrix

$$I_k = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$k \times k$ matrix

Note: I_k is a **square-matrix**.

comment: write I to not confuse with the number one 1

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exercise: prove this

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I_n and I_m are **different** matrices!

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Exercise

Compute AI_3 and I_2A for

$$A = \begin{pmatrix} -8 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix}$$

System of Linear Equations for I_n

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What system of linear equations does I_n represent?

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Consider the augmented matrix $(I_n|\vec{b})$

$$(I_n|\vec{b}) = \left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & b_1 \\ 0 & 1 & \cdots & 0 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_n \end{array}\right)$$

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(I_n|\vec{b}) = \begin{array}{c} \begin{array}{cccc|c} x_1 & x_2 & & x_n & \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & b_1 \\ 0 & 1 & \cdots & 0 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_n \end{array}\right) \end{array} \implies \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \\ b_n \end{bmatrix} = \vec{b}
$$

And so we see that $\vec{x}$ solves $(I_n|\vec{b})$ if and only if $\vec{x} = \vec{b}$

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And so we see that $\vec{x}$ solves $(I_n|\vec{b})$ if and only if $\vec{x} = \vec{b}$

Solving $(I_n|\vec{b})$ Using Matrix Multiplication

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from which we can conclude that $\vec{x}$ solves $(I_n|\vec{b})$ if and only if

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A is $n \times n$

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Let A be the coefficient matrix of a system of n linear equations with n variables. Then A is a square matrix. Let R be the RREF of A.

1. *If $R = I_n$, then the system has a unique solution* for all $(A|\vec{b})$
2. *If $R \neq I_n$ then $rk(A) < n$ and the system has either 0 or infinitely many solutions.*

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Recall: we say that two matrices are **row equivalent** if one can be obtained from the other from a series of row operations. So the condition $R = I_n$ can be rephrased as A is row equivalent to I_n and $R \neq I_n$ can be rephrased as A is *not* row equivalent to I_n.

If A is row equivalent to I_n then $(A|\vec{b})$ has a unique solution for all $\vec{b}$.

Proof

① $(A|\vec{b}) \xrightarrow{\text{row reduce}} (R|\vec{c}) = (I_n|\vec{c})$

x solves $(A|\vec{b})$ iff $\vec{x}$ solves $(I_n|\vec{c})$ iff $\underline{\vec{x} = \vec{c}}$

② $(A|\vec{b}) \xrightarrow{\text{row reduce}} (R|\vec{c})$

# of free variables $= n - rk(R)$
so if $rk(R) < n \Rightarrow$ # of free variables > 0

if $rk(R) = n$: By exhausting when the leading 1's can be

$R = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & 1 \end{bmatrix} = I_n$.

So, if $R \neq I_n$ then $rk(R) \neq n$ & so $rk(R) < n$

so R must have $< n$ leading ones but n rows so by exhaustion R must have a row of zeros.

$R = \begin{bmatrix} & & \\ 0 & \cdots & 0 \end{bmatrix}$ & so $(R|\vec{c}) = \left[\begin{array}{ccc|c} & & & \\ 0 & \cdots & 0 & c_n \end{array}\right]$ if $c_n \neq 0$ no solutions

if $c_n = 0 \Rightarrow$ then must be a free variable $\Rightarrow \infty$ solutions

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Division is defined as the opposite of multiplication. So, it is better to think of it not as it's own operation but as a type of multiplication. That is:

"dividing by a" is the same as "multiplying by $\frac{1}{a}$"

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Conclusion

To solve $ax = b$ it is best to think about *multiplying by the multiplicative inverse* of a then to think about dividing by a.

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I behaves like 1

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So, following this, for a matrix A, the multiplicative inverse is

the *matrix* that, when multiplied by A, is I, the identity matrix

Sanity Check

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$$A\vec{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \implies BA\vec{x} = B\begin{bmatrix} 1 \\ -2 \end{bmatrix} \implies I_2\vec{x} = \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

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$$\implies \vec{x} = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$$

Invertible Matrices

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CAUTION!!!!!

This definition if for **square** matrices only $(m = n)$.

A is $\underline{m} \times \underline{n}$ matrix
B is $\underline{k} \times \underline{l}$ matrix

Want $\underline{AB}$ & $\underline{BA}$ to make sense

$\underset{m \times l}{\underline{AB}} = \underset{k \times n}{\underline{BA}}$

$m = l = k = n$

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CAUTION!!!!!

This definition if for **square** matrices only ($m = n$). We can not talk about inverse of non-square matrices!

Solving Matrix Equations with Invertible Matrices

Theorem

If A in an invertible matrix then

$$A\vec{x} = \vec{b} \iff \vec{x} = A^{-1}\vec{b}.$$

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In particular, there is exactly one solution to the augmented matrix $(A|\vec{b})$ for all $\vec{b}$.

$(\Rightarrow)$ $A\vec{x} = \vec{b}$ multiply both sides by A^{-1}: $A^{-1}A\vec{x} = A^{-1}b$

$\rightarrow I_n x = A^{-1}b$

$x = A^{-1}b$

$(\Leftarrow)$ $\vec{x} = A^{-1}b$ then $A\vec{x} = AA^{-1}\vec{b} = I_n\vec{b} = \vec{b}$

Invertibility Theorem

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1. $A\vec{x} = \vec{b}$ *has a unique solution for every* $\vec{b}$

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Let A be an $n \times n$ matrix. The the following are equivalent

1. $A\vec{x} = \vec{b}$ *has a unique solution for every* $\vec{b}$
2. $A\vec{x} = 0$ *has a unique solution*

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Let A be an $n \times n$ matrix. The the following are equivalent

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(1) => (2) => (3) => (4) => (5) => (1)

(1) => (2) If $A\bar{x} = \vec{b}$ has a solution for all $\vec{b}$ then in particular for $\vec{b} = 0$

(2) => (3) $A\bar{x} = 0$ always has a solution of $\vec{x} = 0$

so $Ax = 0$ has 1 solution or ∞ solutions.

If it has ∞ many solution => A has a free variable.

If A has a free variable then $rk(A) = n - \#\text{free variable} < n$

More Work Space

(2) → (3) continued so working backwards if rk A < n then A has a free variable & thus infinitely many solutions to $Ax=0$. Hence impossible since we are (2) $Ax=0$ has 1 solution

(3) ⇒ (4) $A \Rightarrow R$ R is a square $n\times n$ matrix with n leading 1s. and so by exhaustion

$$R = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = I_n$$

(5) ⇒ 1 If A is invertible then

$Ax=b \Rightarrow x = A^{-1}b$ and so unique solution.

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To prove the (4) $\implies$ (5), we need to know more about the inverse.

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5. *AB is invertible with $(AB)^{-1} = B^{-1}A^{-1}$.* — big one!

(1) Note that $I_n I_n = I_n \Rightarrow I_n^{-1} = I_n$

(2) let A be an invertible matrix & assume

$AB = BA = I$ but also $AC = CA = I$

$B = IB = CAB = CI = C$

More Work Space

③ Want to show $(A^{-1})^{-1} = A$

$B = (A^{-1})^{-1}$ is the matrix such that $A^{-1}B = BA^{-1} = I$

We note that $A = B$ satisfies this & by the uniqueness is the only matrix that satisfies this

④ $(cA)^{-1} = \frac{1}{c}A^{-1}$ check: $(cA)\cdot(\frac{1}{c}A^{-1}) = c\cdot\frac{1}{c}\cdot A\cdot A^{-1}$

$= 1 \cdot I = I$

$$(AB)^{-1} \neq A^{-1}B^{-1}$$

⑤ $(AB)^{-1} = B^{-1}A^{-1}$

check that $(B^{-1}A^{-1})\cdot(AB) = B^{-1}(A^{-1}A)B$

$= B^{-1}(IB) = B^{-1}B = I$

Elementary Matrices

Recall that the elementary row operation are

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Definition

We say a matrix is an **elementary matrix** if it can be obtained by one elementary row operation performed on the identity matrix.

Examples of Elementary Matrices

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$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2+4R_1} \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E$$

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$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{-2R_1} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E$$

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Exercise: prove this

$$R_2+4R_1 \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \leftarrow R_2-4R_1$$

$$R_3 \Leftrightarrow R_1 \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \leftarrow R_1 \Leftrightarrow R_3$$

$$-2R_1 \rightarrow \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \leftarrow -\frac{1}{2}R_1$$

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$R_2 + 4R_1$

$$\xrightarrow{} \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overbrace{\begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 6 & -1 & 11 & 18 \\ 1 & 4 & 4 & 0 \end{pmatrix}}^{R_2 + 4R_1}$$

$2 + 4 \cdot 1$ $\qquad$ $3 + 4 \cdot 2$

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Theorem

Two matrices A and B are row equivalent if and only if there is a series of elementary matrices such that

$$B = E_k E_{k-1} \cdots E_2 E_1 A$$

(4) $\implies$ (5)

Theorem

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The most important equivalence

$(2) \iff (5)$

$(4) \Rightarrow (5)$ RREF of A is $I_n \Rightarrow A$ & I_n are row equivalent

$\Rightarrow \exists\ E_1, E_2 \ldots E_k$ such that $I_n = E_k E_{k-1} \cdots E_1 A$

$\Rightarrow A = (E_k E_{k-1} \cdots E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1}$ aside

So we have a B ($B = E_k E_{k-1} \cdots E_1$) such that $BA = I_n$

& so A is invertible. (check $AB = I_n$)

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performing the row operations on I

Conclusion

To find the inverse of A, it is enough to apply row operation that reduce A to I on I itself.

Algorithm for Finding the Inverse

1. Augment the $n \times n$ matrix A with I_n: $(A|I_n)$

$$(A|I_n) = \left(\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & & & 0 \\ \vdots & & & & & 1 & & \\ & & & & & & \ddots & \\ a_{n1} & \cdots & \cdots & a_{nn} & 0 & & & 1 \end{array}\right)$$

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1. Augment the $n \times n$ matrix A with I_n: $(A|I_n)$
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4. If $R \neq I_n$ then A is not invertible.

Finding the Inverse

Find the inverse of the matrix

$$A = \begin{pmatrix} -1 & 3 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

$$(A \mid I_n) = \left(\begin{array}{ccc|ccc} -1 & 3 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \quad -1R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & -3 & 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2 - R_1 \\ R_3 - 2R_1 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & -3 & 1 & -1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 1 & 0 \\ 0 & 7 & -2 & 2 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & -3 & 1 & -1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{array}\right) \quad \tfrac{1}{3}R_2 \quad \left(\begin{array}{ccc|ccc} 1 & -3 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1/3 & 1/3 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{array}\right) R_1$$

$R_1 \to R_1 + 3R_2$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1/3 & 1/3 & 0 \\ 0 & 0 & -2 & 2/3 & -2/3 & 1 \end{array}\right)$$

$-\tfrac{1}{2}R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1 & -1/3 & 2/6 & -1/2 \end{array}\right)$$

$R_1 - R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & -1/6 & 1/2 \\ 0 & 1 & 0 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1 & -1/3 & 2/6 & -1/2 \end{array}\right)$$

Warning: There may be computational errors

$$\underline{\begin{pmatrix} -1 & 3 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}^{-1}} = \underline{\begin{pmatrix} 1/3 & -1/6 & 1/2 \\ 1/3 & 1/3 & 0 \\ -1/3 & 2/6 & -1/2 \end{pmatrix}}$$

Check: multiplying these matrices gives $\underline{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$